Priority weights in claims problems*

Karol Szwagrzak† and Albin Erlanson‡

October 10, 2014

Abstract

We revisit the claims problem (O'Neill, 1982; Aumann and Maschler, 1985) where a group of individuals have claims on a resource but there is not enough of it to honor all of the claims. We characterize the solutions satisfying consistency, composition up, and claims truncation invariance. These solutions are specified by a pair of weights for each individual; the first weight determines a priority class for the individual, the second one determines how favorably she is treated within this class. This characterization holds for the discrete version of the claims problem, where the resource is available in indivisible units (Moulin, 2000). Moreover, it extends to a generalized version of the claims problem where there are multiple resources and individuals have claims on each of these. By duality, our results yield the characterization of the solutions satisfying consistency, composition down, and minimal rights first.

Keywords: Consistency; Composition up; Claims truncation invariance; Indivisible resources; Duality
JEL classification: D70, D63, D71

---

*We thank William Thomson and Juan D. Moreno-Ternero for invaluable discussions at early stages of this work. Further detailed comments by William Thomson substantially improved the presentation of the material herein. We are also grateful to Tommy Andersson, Michel Le Breton, Paolo Piacquadio, Rafael Treibich, José Zarzuelo, and to the participants in the Third International Workshop in Game Theory at the University of Southern Denmark for suggestions. Albin Erlanson gratefully acknowledges financial support from the European Research Council and the Jan Wallander and Tom Hedelius Foundation.

†University of Southern Denmark. E-mail: karol.s.flores@gmail.com

‡Bonn University. E-mail: aerlanso@uni-bonn.de

1 Introduction

We revisit the classical claims problem (O'Neill, 1982; Aumann and Maschler, 1985) where a group of individuals have claims on a resource but there is not enough of it to honor all of the claims. Examples include the distribution of the liquidation value of a bankrupt firm among its creditors and the division of the cost of a public project or a tax burden among individuals with different incomes (Young, 1987, 1988).

Our goal is to evaluate the potential solutions - distribution methods - for the claims problem on the basis of three classical and normatively compelling properties:

- *Consistency* (Aumann and Maschler, 1985; Young, 1987) specifies that if a distribution is considered desirable, then it should be considered desirable when restricted to each subgroup of individuals.[1]

- *Composition up* (Young, 1988; Moulin, 1987) specifies that, upon an increase in the endowment, the solution can recommend the distribution in two equivalent ways: (i) Apply the solution directly to distribute the larger endowment. (ii) Apply the solution to distribute the initial endowment and, thereafter, apply it again to allocate the increment according to the outstanding claims.[2]

- *Claims truncation invariance* (Dagan, 1996) specifies that the excess of claims over the endowment should be omitted from consideration.[3] As expressed by Aumann and Maschler (1985) in the bankruptcy context, "any amount of debt to one person that goes beyond the entire estate might well be considered irrelevant; you cannot get more than there is".

Besides being the subject of a growing literature (for surveys, see Thomson, 2003, 2014; Moulin, 2002), all three properties are anchored in Talmudic principles; see Aumann and Maschler (1985) for consistency and Dagan (1996) for composition up and claims truncation invariance.

We characterize the family of solutions satisfying consistency, composition up, and claims truncation invariance. These solutions are specified by a pair of weights for each individual; the first weight determines a priority class for the individual,

[1]Within claims problems, consistency has been a central property in the study of the most important classes of solutions (Young, 1987; Moulin, 2000; Chambers, 2006; Kaminski, 2006; Stovall, 2014b,a). Beyond claims problems, consistency has been one of the most thoroughly studied properties in the axiomatic resource allocation literature. See Thomson (2011) for a survery on consistency.

[2]Moulin (1987) refers to the property as "path independence". A conceptually similar property in Nash bargaining is "step by step negotiations" (Kalai, 1977).

[3]Dagan (1996) refers to the property as "independence of irrelevant claims".

the second one determines how favorably she is treated within this class. In the highest priority class, resources are distributed in proportion to the second weight parameter. If each individual in the highest class is able to receive her claim, the distribution moves on to the next priority class where the remaining resources are distributed, again, in proportion to the second weight parameter.

The classical "constrained equal awards solution"[4] is a member of this family. It corresponds to the case where all individuals have equal weights. On the other end of the equity spectrum, sequential priority solutions correspond to the case where all individuals have a different first weight parameter.

The characterization holds for the discrete version of the claims problem studied by Moulin (2000) where the resource comes in indivisible units. Moreover, it also extends to a generalized version of the claims problem where there are multiple resources and individuals have claims on each of these. In this case, resources may be divisible while others may come in indivisible units. By duality, our results yield the characterization of all consistent solutions satisfying the "minimal rights first" and "composition down" (Moulin, 2000).

Outline Section 2 deals with the classical claims problem. It contains the characterization of the "weighted priority solutions" on the basis of consistency, composition up, and claims truncation invariance. It also presents a characterization of its "dual" class of solutions. Section 3 deals with the multi-resource extension of the claims problem. It contains the proof that our main results extend to this more general setting without changes.

2 Classical claims problems

2.1 Model

A resource that may be divisible or come in indivisible units is to be allocated among a group of individuals drawn form a finite set A.[5] Let $\mathcal{N}$ denote the collection of subsets of A. For each group of individuals $N \in \mathcal{N}$, a **claims problem** is the pair $(c, e) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq e$. For each $N \in \mathcal{N}$, let $\mathcal{P}^N$ denote the claims problems involving the individuals in N. An **allocation** for the claims problem $(c, e) \in \mathcal{P}^N$ is a profile $z \in \mathbb{R}^N_+$ such that $\sum_N z_i = e$ and, for each $i \in N$,

[4] The solution is sometimes referred to as the "uniform gains method" (Moulin, 2000, 2002).

[5] The basic mathematical notation is as follows: Let $\{Y_i\}_{i \in I}$ be a family of sets Y_i indexed by I. Let $Y^I \equiv \times_{i \in I} Y_i$. For each $y \in Y^I$ and each $J \subseteq I$, we denote by y_J the projection of y onto Y^J. If $x, y \in \mathbb{R}^I$, then $x \geq y$ means that, for each $i \in I$, $x_i \geq y_i$.

$z_i \leq c_i$. Naturally, if the resource comes in indivisible units, the claims problem (c, e) is assumed to be in $\mathbb{Z}^N_+ \times \mathbb{Z}_+$ and the the allocation z is assumed to be in $\mathbb{Z}^N_+$. Let $Z(c, e)$ denote the collection of all allocations for claims problem (c, e). A **solution** is a function φ recommending an allocation for each possible claims problem: for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$, $\varphi(c, e) \in Z(c, e)$.

2.2 Axioms

The objective of this paper is to investigate the joint implications of the following axioms.

Consistency For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$, each $(c, e) \in \mathcal{P}^N$, and each $i \in N'$, $\varphi_i(c_{N'}, \sum_{i \in N'} \varphi_i(c, e)) = \varphi_i(c, e)$.

Composition up For each $(c, e) \in \mathcal{P}^N$ and each e' such that $e' \geq e$, $x = \varphi(c, e)$ implies $\varphi(c, e') = x + \varphi(c - x, e' - e)$.

For each $(c, e) \in \mathcal{P}^N$, let $c \wedge e \equiv (\min\{c_i, e\})_{i \in N}$.

Claims truncation invariance For each $(c, e) \in \mathcal{P}^N$, $\varphi(c \wedge e, e) = \varphi(c, e)$.

Consistency specifies that the solution recommends allocations "in agreement" across claims problems. For instance, suppose that funds are being distributed in a university. The requirement is that, when reassessed within each department, the distribution is still considered desirable. By construction, the procedure whereby funds are first assigned to departments and are then distributed within each is *consistent*. *Consistency* thus specifies that the way a subgroup divides its portion of the funds only depends on the claims of its members.

Consistency has played a central role in the study of claims problems starting with the fundamental works of Aumann and Maschler (1985), Young (1987, 1988), and in the subsequent developments by Moulin (2000). Beyond claims problems, it has been one of the most thoroughly studied properties in the axiomatic resource allocation literature since it was introduced by Harsanyi (1959) in bargaining problems.[6] Balinski (2005) justifies this emphasis, arguing that *consistency* is a fundamental component of equitable resource allocation.[7] Thomson (2012) provides further normative arguments for it.

[6] See Thomson (2011) for a survery on *consistency*.

[7] Balinski refers to *consistency* as "coherence".

Composition up specifies that increases in the endowment should be allocated with the same solution as the initial endowment. Upon distributing the original endowment, we are left with "residual claims": each individual's claim is revised down by the portion of original endowment she received. These residual claims then ought to be used as claims to distribute the increase. *Composition up* appears in the characterizations of the "equal sacrifice taxation methods" (Young, 1988), the "constrained equal awards rule" (Dagan, 1996), and in that of the asymmetric rationing methods of Moulin (2000).

2.3 Weighted priority solutions

We provide two equivalent descriptions of our proposed class of solutions.

Description 1. A solution φ is a **weighted priority solution** if, for each $i \in A$, there is a positive integer n_i and a positive number w_i such that, for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

$$\varphi(c, e) = \arg\max\{\textstyle\sum_{i \in N} n_i x_i + w_i \ln(1 + x_i/w_i) : x \in Z(c, e)\}.$$

If the resource comes in indivisible units, for each pair $i, j \in A$, $n_i \neq n_j$.

Intuitively, individuals with the highest n_i are given priority because their marginal return in the above optimization problem is greater. Individuals with lower values of n_i receive the resource only once the highest n_i individuals present are awarded their claims. We can thus think of the n_i parameters as defining *priority classes*. Among those individuals with the highest n_i, resources are distributed in proportion to w_i conditional on no individual receiving more than her claim. Thus, the higher her n_i and w_i parameters, the better for that individual.

The following characterization of the weighted priority solutions and its extension to the generalized claims problems in Section 3 are our central contributions.

Theorem 1. *A solution satisfies consistency, composition up, and claims truncation invariance if and only if it is a weighted priority solution.*

The classical "constrained equal awards solution" (for divisible resources) corresponds to the case where all individuals have the same parameters n_i and w_i. The more conventional definition of the constrained equal awards is as follows: φ it the **constrained equal awards (CEA) solution** if, for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

for each $i \in N$, $\varphi_i(c, e) = \min\{c_i, \lambda\}$ where λ is such that $\sum_N \min\{c_i, \lambda\} = e$.

To see that our optimization based-definition is equivalent to the standard definition of CEA above, note that this solution recommends the "Lorenz dominant" allocation for each claims problem.[8] Now, if a Lorenz dominant allocation exists, then, using a classical result due to Hardy, Littlewood and Polya, it also maximizes any separable and symmetric concave function (see Schmeidler, 1979). Thus, given positive numbers n_0 and w_0, CEA obtains by maximizing $\sum_{i \in N} n_0 z_i + w_0 \ln(1 + z_i/w_0)$ over the collection feasible allocations.

Moreover, if all individuals have the same parameter n_i and resources are divisible, the weighted priority solutions reduce to the following class of well known solutions. A solution φ is a **generalized constrained equal awards (GCEA) solution** if there is $w \in \mathbb{R}^A_{++}$ and, for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

$$\text{for each } i \in N,\ \varphi_i(c, e) = \min\{c_i, w_i\lambda\} \text{ where } \lambda \text{ is such that } \sum_N \min\{c_i, w_i\lambda\} = e.$$

Moulin (2002) refers to GCEAs as "weighted gains methods". To recast these solutions in the format of our weighted priority solutions it suffices to equalize the first set of parameters across all agents: given a positive integer n_0, a GCEA obtains by maximizing $\sum_{i \in N} n_0 z_i + w_i \ln(1 + z_i/w_i)$ over the collection of feasible allocations.

We now address the question of what extra condition on the class of weighted priority solutions characterizes the class of GCEA solutions. As the solutions themselves, the requirement is meaningful only when resources are divisible since it specifies that, whenever the endowment is positive and an individual has a positive claim, she is not left with nothing.

Positive awards Suppose that the resource is divisible. For each $(c, e) \in \mathcal{P}^N$ and each $i \in N$ such that $c_i > 0$, $e > 0$ implies $\varphi_i(c, e) > 0$.

Positive awards is a *minimal* inequality aversion property.

Corollary 1. *Suppose that the resource is divisible. A solution satisfies consistency, composition up, claims truncation invariance, and positive awards if and only if it is a generalized constrained equal awards solution.*

See the Appendix for a proof of Corollary 1. Clearly, CEA is the only GCEA solution satisfying the following axiom:

[8] This can be deduced from the results in Schummer and Thomson (1997). See also Theorem 19 in Thomson (2014) and the references therein.

Equal treatment of equals: Suppose that the resource is divisible. For each $(c, e) \in \mathcal{P}^N$ and each pair $i, j \in N$ such that $c_i = c_j$, $\varphi_i(c, e) = \varphi_j(c, e)$.

In fact, Dagan (1996) proved that CEA is the only solution satisfying *composition up*, *claims truncation invariance*, and *equal treatment of equals*. Thus, *consistency* is redundant in the following corollary.

Corollary 2. *Suppose that the resource is divisible. A solution satisfies consistency, composition up, claims truncation invariance, and equal treament of equals if and only if it is the constrained equal awards solution.*

Each weighted priority solution can also be defined in similar ways to those used for CEA and the GCEA solutions. Firstly, partition individuals into priority classes. The allocation is computed by firstly distributing the resource among the individuals in the highest priority class using a GCEA solution. Thereafter, if an amount of the resource remains, it is distributed among the individuals in the second highest priority class, again using a GCEA solution, and so on. If the resource comes in indivisible units, the priority classes consist of a single individual so there are exactly $|A|$ distinct priority classes. We are now ready to formally present our second description of the weighted priority solutions.

Description 2. A solution φ is a **weighted priority solution** if there is a partition of A into $n \leq |A|$ cells, $A_1, \ldots, A_n$, a $w \in \mathbb{R}^A_{++}$ and, for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

for each $i \in N \cap A_1$, $\varphi_i(c, e) = \min\{c_i, w_i\lambda_1\}$ where λ_1 is chosen so that

$$\textstyle\sum_{N\cap A_1} \min\{c_i, w_i\lambda_1\} = \min\{e, \sum_{N\cap A_1} c_i\};$$

for each $i \in N \cap A_2$, $\varphi_i(c, e) = \min\{c_i, w_i\lambda_2\}$ where λ_2 is chosen so that

$$\textstyle\sum_{N\cap A_2} \min\{c_i, w_i\lambda_2\} = \min\{e - e', \sum_{N\cap A_2} c_i\}$$

and $e' \equiv \min\{e, \sum_{N\cap A_1} c_i\}$ is the amount distributed among $N \cap A_1$;

$$\vdots$$

If the resource comes in indivisible units, $n = |A|$.

Above, the partition of A into $A_1, \ldots, A_n$ can be interpreted as sorting individuals in A into priority classes: individuals in A_1 have the highest priority, those in A_2 have the second highest priority, and so forth.

2.4 Logical independence

The axioms in Theorem 1 are logically independent. For simplicity, we provide examples illustrating this independence in the case where resources are divisible.

There is a *consistent* solution satisfying *composition up* that is not *claims truncation invariant*. It is the "proportional solution": for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$, $Pro(c, e) \equiv \lambda c$ where $\lambda \in \mathbb{R}_+$ is chosen so that $\lambda \sum_N c_i = e$. Now suppose that $c'_i > c_i \geq e$. Then, $Pro_i(c'_i, c_{-i}, e) > Pro_i(c, e)$, contradicting *claims truncation invariance*.

There is a *consistent* and *claims truncation invariant* solution that does not satisfy *composition up*. For example, consider the solution ϕ defined as follows: for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$, $\phi(c, e) \equiv Pro(c \wedge e, e)$. To see this solution does not satisfy *composition up*, let $N \equiv \{i, j\} \in \mathcal{N}$ and consider $(c, e) \in \mathcal{P}^N$ such that $c_i = 1$, $c_j = 3$, $e = 1$, and $e' = 2$. Then, $\phi(c, e) = (\frac{1}{2}, \frac{1}{2})$ and $\phi(c, e') = (\frac{2}{3}, \frac{4}{3})$. However, $\phi(c, e) + \phi(c - (\frac{1}{2}, \frac{1}{2}), e - e') = (\frac{1}{2}, \frac{1}{2}) + (\frac{1}{3}, \frac{2}{3}) = (\frac{5}{6}, \frac{7}{6}) \neq \phi(c, e')$, establishing that ϕ does not satisfy *composition up*.

There is a *claims truncation invariant* solution satisfying *composition up* that is not *consistent*. For example, let $A = \{1, 2, 3\}$ and consider the solution ϕ defined as follows: for each $(c, e) \in \mathcal{P}^A$, $\phi(c, e) = x$ if $x_1 = \min\{c_1, e\}$, $x_2 = \min\{c_2, e - x_1\}$, and $x_3 = e - x_1 - x_2$; for each $(c, e) \in \mathcal{P}^{\{1,2\}}$, $\phi(c, e) = x$ if $x_1 = e - x_2$ and $x_2 = \min\{c_2, e\}$; for each $(c, e) \in \mathcal{P}^{\{1,3\}}$, $\phi(c, e) = x$ if $x_1 = e - x_3$ and $x_3 = \min\{c_3, e\}$; for each $(c, e) \in \mathcal{P}^{\{2,3\}}$, $\phi(c, e) = x$ if $x_2 = e - x_3$ and $x_3 = \min\{c_3, e\}$. Now let $(c, e) \in \mathcal{P}^A$ be such that $c = (1, 1, 1)$ and $e = 1$ and let $x \equiv \phi(c, e)$. Then, $x = (1, 0, 0)$. However, $\phi(c_{1,2}, x_1 + x_2) = (0, 1)$. That is, $\phi(c_{1,2}, x_1 + x_2) \neq x_{\{1,2\}}$ contradicting *consistency*.

2.5 Duality

Duality, as it is understood in claims problems, is the idea that the distribution of gains is mirrored by the distribution losses. That is, the distribution of the endowment (e) and the distribution of the shortfall ($\sum c_i - e$) are linked; a solution for the first problem defines a solution for the second problem and conversely.

For each solution φ, its **dual** is the solution φ^d defined by

$$\text{for each } N \in \mathcal{N} \text{ and each } (c, e) \in \mathcal{P}^N,\ \varphi^d(c, e) = c - \varphi(c, \textstyle\sum_{i \in N} c_i - e).$$

Note that the dual of the dual solution is original solution, $\varphi^{dd} = \varphi$.

Two properties are dual if, whenever a solution satisfies one of them, its dual satisfies the other. The dual of *composition up* (Moulin, 2000) requires that, upon a decrease in the endowment, one should be able to calculate the allocation in two equivalent ways: (i) Apply the solution directly, ignoring the recommendations for the original amount. (ii) Apply the solution using the allocation for the original endowment as if it was the claims profile.

Composition down For each $(c,e) \in \mathcal{P}^N$ and each $e' \leq e$, $x = \varphi(c,e)$ implies $\varphi(c,e') = \varphi(x,e')$.

To define the dual of *claims truncation invariance* we first need to propose a notion of minimal entitlements.[9] For each $N \in \mathcal{N}$, each $(c,e) \in \mathcal{P}^N$, and each $i \in N$, the **minimal right of individual $\boldsymbol{i}$ in $\boldsymbol{(c,e)}$** is the difference between the endowment and the sum of the claims of the other individuals if this difference is positive and zero otherwise,

$$m_i(c,e) \equiv \max\{0, e - \textstyle\sum_{N\setminus\{i\}} c_j\}.$$

Let $m(c,e) \equiv (m_i(c,e))_{i\in N}$. The dual property of *claims truncation invariance* specifies that a solution can be applied to a claims problem in two equivalent ways: either directly or by first attributing to each individual her minimal right and then distributing the remaining endowment according to the outstanding claims.

Minimal rights first For each $(c,e) \in \mathcal{P}^N$ and each $i \in N$,

$$\varphi(c,e) = m(c,e) + \varphi\left(c - m(c,e), e - \textstyle\sum_{i\in N} m_i(c,e)\right).$$

Lemma 1. *(i) If a solution is consistent, then so is its dual. (ii) Composition up and composition down are dual properties. (iii) Claims truncation invariance and the minimal rights first are dual properties.*

We omit the straightforward and well known proof. Theorem 1 and Lemma 1 yield a characterization of the class of *consistent* solutions satisfying *composition down* and *minimal rights first*.

Theorem 2. *A solution satisfies consistency, composition down, and minimal rights first if and only if it is the dual of a weighted priority solution.*

By Lemma 1, the dual of a solution satisfying the properties in Theorem 2 satisfies *consistency*, *composition up*, and *claims truncation invariance*. Thus, by Theorem 1, the solution is the dual of a weighted priority solution.

[9] The dual of *claims truncation invariance* was first discussed by Curiel et al. (1987).

2.6 Proof of Theorem 1

Let φ denote a solution that satisfies *composition up*, *consistency*, and *claims truncation invariance*.

Preliminary lemmas

Lemma 2. *For each $N \in \mathcal{N}$, there is $r \in \mathbb{R}^N_+$ such that $\sum_N r_i = 1$ and such that, for each $(c, e) \in \mathcal{P}^N$, if $e \leq \min_{i \in N} c_i$, then $\varphi(c, e) = er$. Moreover, if the resource comes in indivisible units, there is $i \in N$ such that $r_i = 1$.*

Proof. Let $N \in \mathcal{N}$ and let $(\bar{c}, \bar{e}) \in \mathcal{P}^N$ be such that $\bar{e} = 1 \leq \min_{i \in N} \bar{c}_i$ and let $r \equiv \varphi(\bar{c}, \bar{e})$. Note that $\sum_N r_i = 1$; if the resource comes in indivisible units, feasibility requires there is $i \in N$ such that $r_i = 1$ since exactly one individual will receive the single unit. Let $(c, e) \in \mathcal{P}^N$ be such that $e \leq \min_{i \in N} c_i$. We distinguish three cases:

Case 1. $e = 1$. By *claims truncation invariance*, $\varphi(c, e) = \varphi(\bar{c}, e) = r$.

Case 2. $e < 1$. (This case does not apply if the resource comes in indivisible units.) By *claims truncation invariance*, $\varphi(c, e) = \varphi(\bar{c}, e)$. Thus, assume, without loss of generality by *claims truncation invariance* that $1 \leq \min_{i \in N} c_i$.

- Suppose that $e = \frac{1}{2}$ and let $x \equiv \varphi(c, e)$. Then, by Case 1 and *composition up*, $r = \varphi(c, 1) = x + \varphi(c - x, 1 - \frac{1}{2})$. Then, by *claims truncation invariance*, $\varphi(c - x, 1 - \frac{1}{2}) = x$. Thus, $r = 2x$. Thus, $\varphi(c, \frac{1}{2}) = x = \frac{1}{2}r$.
- Suppose that $e = \frac{1}{4}$ and let $x \equiv \varphi(c, e)$. Then, by *composition up*, $\frac{1}{2}r = \varphi(c, \frac{1}{2}) = x + \varphi(c - x, \frac{1}{2} - \frac{1}{4})$. Then, by *claims truncation invariance*, $\varphi(c - x, \frac{1}{2} - \frac{1}{4}) = x$. Thus, $\frac{1}{2}r = 2x$. Thus, $\varphi(c, \frac{1}{4}) = x = \frac{1}{4}r$.

Continuing in this way we can show that for each pair of natural numbers m and n such that $m \leq 2^n$, $\varphi(c, \frac{m}{2^n}) = \frac{m}{2^n}r$. Moreover, by *composition up*, $\varphi(c, \cdot)$ is continuous.[10] Thus, for each $e < 1$, $\varphi(c, e) = er$.

[10] To see that *composition* implies $\varphi(c, \cdot)$ is continuous note that, for each $\varepsilon > 0$,

$$\|\varphi(c, e+\varepsilon) - \varphi(c, e)\| = \|[\varphi(c, e) + \varphi(c - \varphi(c, e), \varepsilon)] - \varphi(c, e)\| = \|\varphi(c - \varphi(c, e), \varepsilon)\| \leq |N|\varepsilon, \text{ and}$$
$$\|\varphi(c, e-\varepsilon) - \varphi(c, e)\| = \|\varphi(c, e-\varepsilon) - [\varphi(c, e-\varepsilon) + \varphi(c - \varphi(c, e-\varepsilon), \varepsilon)]\| = \|\varphi(c - \varphi(c, e-\varepsilon), \varepsilon)\| \leq |N|\varepsilon.$$

Case 3. $e > 1$. Let n be the largest integer such that $n \leq e$. By *composition up*, *claims truncation invariance*, and Case 1,

$$\begin{aligned}\varphi(c,e) &= r + \varphi(c-r, e-1)\\ &= 2r + \varphi(c-2r, e-2)\\ &\vdots\\ &= nr + \varphi(c-nr, e-n).\end{aligned}$$

By Case 2, $\varphi(c-nr, e-n) = (e-n)r$. Thus, $\varphi(c,e) = nr + (e-n)r = er$. ∎

Lemma 3. *There is* $\{(n_i, w_i) \in \mathbb{Z}_{++} \times \mathbb{R}_{++} : i \in A\}$ *such that, for each* $N \in \mathcal{N}$ *and each* $(c,e) \in \mathcal{P}^N$ *such that* $e \leq \min_{i\in N} c_i$,

$$\varphi(c,e) = \arg\max\{\textstyle\sum_{i\in N} n_i x_i + w_i \ln(1 + x_i/w_i) : x \in Z(c,e)\}.$$

If the resource comes in indivisible units, for each pair $i, j \in A$, $n_i \neq n_j$.

Proof. By Lemma 2, for each $(c,e) \in \mathcal{P}^A$ such that $0 < e \leq \min_{i\in A} c_i$ there is $r^1 \in \mathbb{R}^A_+$ such that $\sum_{i\in A} r^1_i = 1$ and $\varphi(c,e) = r^1 e$. Let $A_1 \equiv \{i \in A : r^1_i > 0\}$. By Lemma 2, for each $(c,e) \in \mathcal{P}^{A\setminus A_1}$ such that $e \leq \min_{i\in A\setminus A_1} c_i$ there is $r^2 \in \mathbb{R}^{A\setminus A_1}_+$ such that $\sum_{i\in A\setminus A_1} r^2_i = 1$ and $\varphi(c,e) = r^2 e$. Let $A_2 \equiv \{i \in A \setminus A_1 : r^2_i > 0\}$. Continuing in this way, define $r^1, r^2, \ldots, r^h$ and $A_1, A_2, \ldots, A_h$ so that $\cup_{l=1}^h A_l = A$. Note that the sets $A_1, A_2, \ldots, A_h$ partition A and that, for each $j \in \{1, \ldots, h\}$, $r^j \in \mathbb{R}_+^{A\setminus \cup_{l=0}^{j-1} A_l}$ with the convention that $A_0 = \varnothing$. Note that, by construction, $h \leq |A|$. For each $i \in A$, let

$$n_i \equiv \begin{cases} |A| & \text{if } i \in A_1,\\ |A|-1 & \text{if } i \in A_2,\\ \vdots & \vdots\\ |A|-h+1 & \text{if } i \in A_h,\end{cases}$$

and, for each $l \in \{1, \ldots, h\}$ and each $i \in A_l$, let $w_i \equiv r^l_i$. Moreover, by Lemma 2, if the resource comes in indivisible units, for each $l \in \{1, \ldots, h\}$, r^l is such that there is $i \in A_l$ such that $r^l_1 = 1$. Thus, since $\sum_{i\in A_l} r^l_i = 1$, in the indivisible resource case each A_l is a singleton. Thus, if the resource is indivisible, for each pair of distinct $i, j \in A$, $n_i \neq n_j$. This concludes the construction of $\{(n_i, w_i) \in \mathbb{Z}_{++} \times \mathbb{R}_{++} : i \in A\}$.

Let $N \in \mathcal{N}$. It remains to prove that, for each $(c,e) \in \mathcal{P}^N$ such that $0 < e \leq \min_{i\in N} c_i$,[11]

$$\varphi(c,e) = a \equiv \arg\max\{\textstyle\sum_{i\in N} n_i x_i + w_i \ln(1 + x_i/w_i) : x \in Z(c,e)\}. \quad (1)$$

[11] If $e = 0$, $Z(c,e) = \{z\}$ where z is a profile of zeros in $\mathbb{R}^N_+$ and there is nothing to prove.

By Lemma 2, there is $r \in \mathbb{R}^N_+$ with $\sum_{i \in N} r_i = 1$ such that

$$\text{for each } e \leq \min_{i \in N} c_i, \quad \varphi(c, e) = re. \tag{2}$$

Let $l \in \{1, \ldots, h\}$ denote the smallest number such that $N \cap A_l \neq \varnothing$. To establish (1), we first prove that

$$\text{for each } i \in N,\ r_i > 0 \text{ implies } i \in A_l \text{ and, for each pair } i, j \in A_l \cap N,\ \frac{r^l_i}{r^l_j} = \frac{r_i}{r_j}. \tag{3}$$

We first show that $\sum_{i \in N \cap A_l} r_i = 1$. Otherwise there is a $k \in \{1, \ldots, h\}$ such that $k > l$ and a $j \in N \cap A_k$ such that $r_j > 0$. Let $M \equiv \cup^h_{g=l} A_g$ and let $(\tilde{c}, \tilde{e}) \in \mathcal{P}^M$ be such that $\tilde{e} = \min_{i \in M} \tilde{c}_i > 0$ and, for each $i \in N$, $\tilde{c}_i = c_i$. By the definition of r^l, $\varphi(\tilde{c}, \tilde{e}) = r^l \tilde{e}$ and, for each $i \in M \setminus A_l$, $r^l_i = 0$. Thus, $\sum_{i \in A_l} \varphi_i(\tilde{c}, \tilde{e}) = \tilde{e}$. Let $\hat{e} \equiv \sum_{i \in A_l \cap N} \varphi_i(\tilde{c}, \tilde{e})$ and note that, because $A_l \cap N \neq \varnothing$ and $r^l_i > 0$ for $i \in A_l \cap N$, $\hat{e} > 0$. Since $N \subseteq M$, by *consistency*, for each $i \in A_l \cap N$, $\varphi_i(c, \hat{e}) = r^l_i \tilde{e} > 0$, for each $i \in N \setminus A_l$, $\varphi_i(c, \hat{e}) = 0$. However, by (2) and the assumption that there is $j \in N \cap A_k$ such that $r_j > 0$, $\varphi_j(c, \hat{e}) = r_j \hat{e} > 0$, a contradiction since $j \in N \setminus A_l$. Thus, for each $j \in N \setminus A_l$, $r_j = 0$. Moreover, for each pair $i, j \in A_l \cap N$, by (2) and *consistency*,

$$\varphi_i(c, \hat{e}) = r^l_i \tilde{e} = r_i \hat{e} \text{ and } \varphi_j(c, \hat{e}) = r^l_j \tilde{e} = r_j \hat{e}.$$

Thus, for each pair $i, j \in A_l \cap N$, $\frac{r^l_i}{r^l_j} = \frac{r_i}{r_j}$. We have thus established (3).

We now use (2) and (3) to prove (1). Let $i, j \in N$. Note that $n_i > n_j$ implies that, for each pair $y_i, y_j \in \mathbb{R}_+$,

$$\frac{\partial}{\partial z_i}[n_i z_i + w_i \ln(1 + z_i/w_i)]_{z_i = y_i} > \frac{\partial}{\partial z_j}[n_j z_j + w_j \ln(1 + z_j/w_j)]_{z_j = y_j}.$$

Thus, in the optimization problem defining a, if $n_i > n_j$, the marginal return of assigning to i is always greater than that of assigning to j. Thus, if $a_i < c_i$ and $n_i > n_j$, $a_j = 0$. Thus, $a_k > 0$ only if $k \in A_l \cap N$ since for each such k, the value of n_k is maximal among all $k \in N$. Thus, by (2) and (3), for each $k \in N \setminus A_l$, $a_k = 0 = \varphi_k(c, e)$.

It remains to prove that, for each $k \in A_l \cap N$, $a_k = \varphi_k(c, e)$. If $\{i\} = A_l \cap N$, then, by the previous result, for each $k \in N \setminus A_l$, $a_k = 0 = \varphi_k(c, e)$. Thus, by feasibility, $a_i = e = \varphi_i(c, e)$. Thus, assume that $|A_l \cap N| \geq 2$ and let $i, j \in A_l \cap N$. Recall this is only possible when the resource is divisible. Thus, $n_i = n_j$. Thus, by the optimality of a in (1), since $e \leq \min_{k\in N} c_k$, this requires[12]

$$\frac{\partial}{\partial z_i}[n_i z_i + w_i \ln(1 + z_i/w_i)]_{z_i=a_i} = \frac{\partial}{\partial z_j}[n_j z_j + w_j \ln(1 + z_j/w_j)]_{z_j=a_j}.$$

or equivalently, $a_i/w_i = a_j/w_j$. By the definition of w_i and w_j, this is equivalent to $\frac{a_i}{a_j} = \frac{r_i^l}{r_j^l} = \frac{r_i}{r_j}$ where the last equality follows from (3). Thus, by (2), for each $i \in A_l \cap N$, $\varphi_i(c, e) = r_i e = a_i$. Altogether, $a = \varphi(c, e)$. ∎

Converse consistency For each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

$$[x \in Z(c, e) \text{ and, for each } \{i, j\} \subseteq N,\ x_{\{i,j\}} = \varphi(c_{\{i,j\}}, x_i + x_j)] \Rightarrow x = \varphi(c, e).$$

Lemma 4. *The weighted priority solutions satisfy consistency, composition up, claims truncation invariace, and converse consistency.*

See the Appendix for a proof of Lemma 4.

Proof of Theorem 1

Let $N \in \mathcal{N}$, $(c, e) \in \mathcal{P}^N$, and $x \equiv \varphi(c, e)$. If $e \leq \min_{i\in N} c_i$, by Lemma 3, there is $\{(n_i, w_i) \in \mathbb{Z}_{++} \times \mathbb{R}_{++} : i \in A\}$ such that

$$x = a \equiv \arg\max\{\textstyle\sum_{i\in N} n_i z_i + w_i \ln(1 + z_i/w_i) : z \in Z(c, e)\}.$$

Thus, assume from here on that $e > \min_{i\in N} c_i$. The rest of the proof consists of showing, again, that $x = a$.

Let ψ denote the weighted priority solution specified by $\{(n_i, w_i) \in \mathbb{Z}_{++} \times \mathbb{R}_{++} : i \in A\}$.[13] Thus, for example, $\psi(c, e) = a$.

Step 1. *Let $I \equiv \{i, j\} \subseteq N$ and $e_I \equiv x_i + x_j$. Then, $a_I = x_I$.*

[12] Recall that in this moment we are assuming the resource is divisible and we may as well take $e < \min_{k\in N} c_k$ to avoid a corner solution and have the equality between the derivatives. However, since this is true for each e such that $e < \min_{k\in N} c_k$ it is also true when $e = \min_{k\in N} c_k$.

[13] That is, for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

$$\psi(c, e) \equiv \arg\max\{\textstyle\sum_{i\in I} n_i z_i + w_i \ln(1 + z_i/w_i) : z \in Z(c, e)\}.$$

Proof. By *consistency*, $x_I = \varphi(c_I, e_I)$ and, by the *consistency* of ψ (Lemma 4), $\psi(c_I, e_I) = a_I$. If $e_I \leq \min\{c_i, c_j\}$, then by Lemma 3, $x_I = a_I$. Thus, assume from here on that $e_I > \min\{c_i, c_j\}$. We will show, again, that $x_I = a_I$. If $e_I = c_i + c_j$, then by feasibility $x_I = a_I$. Thus, from here on, assume also that $e_I < c_i + c_j$. Recursively define the sequences $\{c^k\}_{k\in\mathbb{N}}$, $\{e^k\}_{k\in\mathbb{N}}$, and $\{x^k\}_{k\in\mathbb{N}}$ as follows:

$$\begin{aligned}
c^1 &= c_I \\
e^1 &= \min\{c_i, c_j\} \\
x^1 &= \varphi(c^1, e^1) \\
c^2 &= c^1 - x^1 \\
e^2 &= \min\{e_I - e^1, c_i^2, c_j^2\} \\
x^2 &= \varphi(c^2, e^2) \\
c^3 &= c^2 - x^2 \\
e^3 &= \min\{e_I - e^1 - e^2, c_i^3, c_j^3\} \\
&\vdots
\end{aligned}$$

Note that by construction, for each k, $(c^k, e^k) \in \mathcal{P}^I$ and $e^k \leq \min\{c_i^k, c_j^k\}$. Thus, Lemma 2 applies to each of the claims problems (c^k, e^k). Thus, there is $r \in \mathbb{R}^I_+$ with $r_i + r_j = 1$ such that

$$\text{for each } k, \quad x^k = re^k. \tag{4}$$

Case 1. There is a finite n such that and $e^1 + \cdots + e^n = e$.

By (4) and *composition up*,

$$\varphi(c_I, e^1 + \cdots + e^n) = x^1 + \cdots + x^n = re^1 + \cdots + re^n = r(e^1 + \cdots + e^n) = re. \tag{5}$$

Case 2. There is no finite n such that $e^1 + \cdots + e^n = e$.

The sequence $\{\sum_{k=1}^h e^k\}_{h\in\mathbb{N}}$ is monotone increasing and bounded above by e_I. It thus has a limit e^*. The sequence $\{e^n\}_{n\in\mathbb{N}}$ is monotone decreasing and bounded below by 0. From the Cauchy convergence criterion applied to the convergent sequence $\{\sum_{k=1}^h e^k\}_{h\in\mathbb{N}}$, for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $e^n = \sum_{k=1}^n e^k - \sum_{k=1}^{n-1} e^k < \varepsilon$. Thus,

$$\{e^h\}_{h\in\mathbb{N}} \text{ converges to zero.} \tag{6}$$

Case 2.1. $e_I = e^*$.

By (4), for each n,

$$\varphi(c_I, e^1 + \cdots + e^n) = x^1 + \cdots + x^n \\ = re^1 + \cdots + re^n = r(e^1 + \cdots + e^n). \tag{7}$$

Thus, since $\{\sum_{k=1}^n e^k\}_{n\in\mathbb{N}}$ converges to e^* and $e^* = e_I$, $\varphi(c_I, e_I) = re_I$.

Case 2.2. $e_I > e^*$.

By (4), for each n,

$$\varphi(c_I, e^1 + \cdots + e^n) = x^1 + \cdots + x^n \\ = re^1 + \cdots + re^n = r(e^1 + \cdots + e^n).$$

Thus, since $\{\sum_{k=1}^n e^k\}_{n\in\mathbb{N}}$ converges to e^*,

$$\varphi(c_I, e^*) = re^*. \tag{8}$$

The sequences $\{c_i^n\}_{n\in\mathbb{N}}$ and $\{c_j^n\}_{n\in\mathbb{N}}$ are monotone decreasing and bounded below. They are thus convergent. Let c_i^* and c_j^* denote their respective limits. Thus, because, for each n, $e_I - \sum_{k=1}^n e^k \geq e_I - e^* > 0$ and, by (6),

$$\min\{e_I - \textstyle\sum_{k=1}^{n-1} e^k, c_i^n, c_j^n\} \equiv e^n \quad \overset{n\to\infty}{\to} \quad 0$$

either $c_i^* = 0$ or $c_j^* = 0$. Otherwise, since $\{e_I - \sum_{k=1}^n e^k\}_{n\in\mathbb{N}}$, $\{c_i^n\}_{n\in\mathbb{N}}$, and $\{c_j^n\}_{n\in\mathbb{N}}$ are all monotone decreasing and convergent, $e^n \to \min\{e_I - e^*, c_i^*, c_j^*\} > 0$, contradicting (6). Thus, without loss of generality assume that $c_i^* = 0$. Thus, by (7),

$$c_i - \textstyle\sum_{k=1}^n r_i e^k = c_i - \sum_{k=1}^n x_i^k \quad \overset{n\to\infty}{\to} \quad c_i^* = 0.$$

Thus, by (8), $\varphi_i(c_I, e^*) = r_i e^* = c_i$. By *composition up* and (8),

$$\varphi(c, e_I) = \varphi(c_I, e^*) + \varphi(c_I - \varphi(c_I, e^*), e_I - e^*) \\ = re^* + \varphi(c_I - re^*, e_I - e^*). \tag{9}$$

Thus, $\varphi_i(c, e_I) = r_i e^* = c_i$ and $\varphi_j(c, e_I) = r_j e^* + e_I - e^* = e_I - c_i$ where the last equality follows from feasibility.

Summarizing, in Cases 1 and 2.1 we find that $\varphi(c_I, e_I) = re_I$. Let $\tilde{c}$ denote a profile of claims for i and j such that $\tilde{c} \geq c_I$ and $e_I \leq \min\{\tilde{c}_i, \tilde{c}_j\}$. By Lemmas 2 and 3, $re_I = \varphi(\tilde{c}, e_I) = \psi(\tilde{c}, e_I)$. Now, $Z(\tilde{c}, e_I) \supseteq Z(c_I, e_I)$ and $re_I \in Z(c_I, e_I)$. Thus, re_I also maximizes $\sum_{k\in I} n_k z_k + w_k \ln(1 + z_k/w_k)$ over $Z(c_I, e_I)$. Thus,

$$\varphi(c_I, e_I) = re_I = \psi(c_I, e_I).$$

In case 2.2, $\varphi_i(c_I, e_I) = r_i e^* = c_i$ and $\varphi_j(c_I, e_I) = r_j e^* + e - e^*$. By the same argument for Cases 1 and 2.1, using e^* instead of e_I,

$$\varphi(c_I, e^*) = re^* = \psi(c_I, e^*).$$

Since ψ satisfies *composition up* (Lemma 4),

$$\psi(c_I, e_I) = \psi(c_I, e^*) + \psi(c_I - \psi(c_I, e^*), e_I - e^*).$$

Thus, since $c_i - \psi_i(c_I, e^*) = 0$, $\psi_i(c_I, e_I) = c_i$ and $\psi_j(c_I, e_I) = r_j e^* + e - e^*$. Thus, $x_I = \varphi(c_I, e_I) = \psi(c_I, e_I) = a_I$, as desired. ∎

Step 2. $a = x$.

Proof. By Step 1, for each $I \subseteq N$ with cardinality of two, $x_I = a_I = \psi(c_I, \sum_I x_i)$. Thus, since ψ is *conversely consistent* (Lemma 4), $x = \psi(c, \sum_N x_i) = \psi(c, e)$. ∎

3 Multidimensional claims problems

3.1 Model

A number of divisible and indivisible resources are to be allocated among a group of individuals drawn form the finite set A. Let $\mathcal{N}$ denote the collection of subsets of A. The resource kinds that are available in indivisible units are indexed by I while those that are available in divisible units are indexed by D. Let K denote the union of I and D. Let $\mathcal{C} \equiv \mathbb{R}^D_+ \times \mathbb{Z}^I_+$ denote the space of possible resource profiles.

For every group of individuals $N \in \mathcal{N}$, a (multidimensional) **claims problem** is the pair (C, E) where $C \in \mathcal{C}^N$ and $E \in \mathcal{C}$ are such that $\sum_N C_i \geq E$. For each $N \in \mathcal{N}$, let $\mathcal{Q}^N$ denote the (multidimensional) claims problems involving the individuals in N. An **allocation** for the claims problem $(C, E) \in \mathcal{Q}^N$ is a profile $z \in \mathcal{C}^N$ such that $\sum_N z_i = E$ and, for each $i \in N$, $z_i \leq C_i$. Let $Z(C, E)$ denote the collection of all allocations for claims problem (C, E). A **solution** is a function φ recommending an allocations for each possible claims problem: for each $N \in \mathcal{N}$ and each $(C, E) \in \mathcal{Q}^N$, $\varphi(C, E) \in Z(C, E)$.

Notation For each $N \in \mathcal{N}$, each $(C, E) \in \mathcal{Q}^N$, and each $k \in K$, let C^k and E^k denote the projections of C and E onto the kth coordinates of $\mathcal{C}^N$ and $\mathcal{C}$, respectively. Thus, C^k is in $\mathbb{R}^N_+$ (in $\mathbb{Z}^N_+$ if $k \in I$) and E^k is in $\mathbb{R}_+$ (in $\mathbb{Z}_+$ if $k \in I$). Similarly, for each $x \in Z(C, E)$, let x^k denote the projection of x onto the kth coordinates of $\mathcal{C}^N$.

3.2 Axioms

The axioms in Section 2.2 can be restated for multidimensional claims problems naturally.

Consistency For each pair $N, N' \in \mathcal{N}$ such that $N' \subseteq N$, each $(C,E) \in \mathcal{Q}^N$, and each $i \in N'$, $\varphi_i(C_{N'}, \sum_{i \in N'} \varphi_i(C,E)) = \varphi_i(C,E)$.

Composition up For each $(C,E) \in \mathcal{Q}^N$, each $E' \in \mathcal{C}$ such that $E \leq E'$, $x = \varphi(C,E)$ implies $\varphi(C,E') = x + \varphi(C - x, E' - E)$.

For each $(C,E) \in \mathcal{Q}^N$, let $C \wedge E$ denote the profile in $\mathcal{C}^N$ such that, for each $k \in K$, $(C \wedge E)^k \equiv (\min\{C_i^k, E^k\})_{i \in N}$.

Claims truncation invariance For each $(C,E) \in \mathcal{Q}^N$, $\varphi(C \wedge E, E) = \varphi(C,E)$.

3.3 Weighted priority solutions

For brevity, we only provide one description of our solutions here.

A solution φ is a **weighted priority solution** if, for each $i \in A$ and each $k \in K$, there is $(n_i^k, w_i^k) \in \mathbb{Z}_{++} \times \mathbb{R}_{++}$ such that, for each $N \in \mathcal{N}$ and each $(C,E) \in \mathcal{Q}^N$,

$$\varphi(C,E) = \arg\max\{\textstyle\sum_{k \in K} \sum_{i \in N} n_i^k x_i^k + w_i^k \ln(1 + x_i^k / w_i^k) : x \in Z(C,E)\}.$$

If resource k comes in indivisible units, so $k \in I$, for each pair $i, j \in A$, $n_i^k \neq n_j^k$.

Theorem 3. *A solution satisfies consistency, composition up, and claims truncation invariance if and only if it is a weighted priority solution.*

Note that the weighted priority solutions are obtained by maximizing a function that is separable over both resource kinds and individuals. Thus, since the constraint set in their definition has a product structure with respect to K, recommendations can be computed by solving $|K|$ disjoint maximization problems. In effect, the weighted priority solutions thus "separate" multidimensional claims problems into standard claims problems. The next lemma, establishes that any *consistent* solution satisfying *composition up* has this property.

Before proceeding with the lemma, we formalize precisely how this separation takes place. For each $N \in \mathcal{N}$, each claims problem $(C,E) \in \mathcal{Q}^N$, and each $k \in K$, let ψ^k denote a function mapping (C^k, E^k) into $\{z \in \mathbb{R}_+^N : \sum_{i \in N} z_i = E^k, z \leq C^k\}$

if $k \in D$ and into $\{z \in \mathbb{Z}_+^N : \sum_{i \in N} z_i = E^k, z \leq C^k\}$ if $k \in I$. Let Ψ denote the collection of profiles $\{\psi^k : k \in K\}$ of such functions.

The following Lemma is the key in extending Theorem 1 to multidimensional claims problems, thus proving Theorem 3. Its proof relies on the following property which is implied by *composition up*.

Resource monotonicity For each $(C, E) \in \mathcal{Q}^N$, each $E' \in \mathcal{C}$ such that $E' \geq E$, $\varphi(C, E') \geq \varphi(C, E)$.

Lemma 5. *Let φ denote a consistent solution satisfying composition up. Then, there is a profile $\{\psi^k : k \in K\} \in \Psi$ such that, for each $N \in \mathcal{N}$, and each $(C, E) \in \mathcal{Q}^N$,*

$$\varphi(C, E) = \{\psi^k(C^k, E^k) : k \in K\}.$$

Proof. Let φ denote a *consistent* solution satisfying *composition up*. Then, φ also satisfies *resource monotonicity* as this property is implied by *composition up*. We will first establish Lemma 5 when $N = A$, and then show that it holds for each $N \in \mathcal{N}$. For each $k \in K$ and each $(c, e) \in [\mathcal{C}^A \times \mathcal{C}]^k$ such that $\sum_A c_i \geq e$, let $\psi^k(c, e) = \varphi(C, E)|^k$ where $(C, E) \in \mathcal{Q}^A$ is such that

i. $C^k = c$ and $E^k = e$;

ii. for each $l \in K \setminus \{k\}$ and each $i \in A$, $E^l = 0$ and $C_i^l = 0$.

By construction, $\{\psi^k : k \in K\}$ is in Ψ. We now prove that $\{\psi^k : k \in K\}$ is as claimed in the statement of Lemma 5. Let $k \in K$, $(C, E) \in \mathcal{Q}^A$, and $x \equiv \varphi(C, E)$. Let $(\bar{C}, \bar{E}) \in \mathcal{Q}^A$ be such that

$$[\text{for each } l \in K \setminus \{k\} \text{ and each } i \in A,\ \bar{E}^l = 0 \text{ and } \bar{C}_i^l = 0].$$

Let $y \equiv \varphi(\bar{C}, \bar{E})$. By the definitions of ψ^k and $(\bar{C}, \bar{E})$, $y^k = \psi^k(C^k, E^k)$. By *resource monotonicity*, for each $i \in A$, $y_i^k \leq x_i^k$. Since $\sum_{i \in A} x_i^k = E^k = \bar{E}^k = \sum_{i \in A} y_i^k$, for each $i \in A$, $\varphi_i(C, E)|^k = x_i^k = y_i^k = \psi^k(C^k, E^k)$. We can repeat this argument for each $k \in K$ to conclude that,

$$\text{for each } k \in K \text{ and each } (C, E) \in \mathcal{Q}^A,\ \varphi(C, E)|^k = \psi^k(C^k, E^k).$$

To finish the proof we have to establish the analogous result for each $N \in \mathcal{N}$. Let $N \in \mathcal{N}$ and $(C, E) \in \mathcal{Q}^N$. Let $(\bar{C}, \bar{E}) \in \mathcal{Q}^A$ be such that

$$[\text{for each } k \in K \text{ and each } i \in A \setminus N,\ \bar{E}^k = E^k,\ \bar{C}_i^k = 0, \text{ and for each } i \in N,\ \bar{C}_i = C_i].$$

Let $y \equiv \varphi(\bar{C}, \bar{E})$ and $x \equiv \varphi(C, E)$. By definition, for each $k \in K$ and each $i \in A \setminus N$, $y_i^k = 0$. Thus, $\sum_N y_i = \sum_N x_i$ and recall that, for each $i \in N$, $\bar{C}_i = C_i$. Thus, by *consistency*, for each $i \in N$, $x_i = y_i$. Thus, for each $k \in K$, $\varphi(C, E)|^k = \psi^k(C^k, E^k)$. ∎

Proof of Theorem 3. Let φ denote a *consistent* and *claims truncation invariant* solution satisfying *composition up*. Clearly, *composition up* implies *resource monotonicity*. Thus, by Lemma 5, there is $\{\psi^k : k \in K\} \in \Psi$ such that, for each N, each $(C, E) \in \mathcal{Q}^N$, and each $k \in K$, $\varphi(C, E)|^k = \psi^k(C^k, E^k)$. Note that, for each $N \in \mathcal{N}$, each $k \in K$, and each $(C, E) \in \mathcal{Q}^N$, $(C^k, E^k) \in \mathcal{P}^N$. Moreover, for each $N \in \mathcal{N}$, each $k \in K$, ψ^k is a solution on $\mathcal{P}^N$. Theorem 3 is thus a consequence of Theorem 1. ∎

References

Aumann, R. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory 36*(2), 195–213.

Balinski, M. (2005). What is just? *American Mathematical Monthly 112*(5), 502–511.

Chambers, C. (2006). Asymmetric rules for claims problems without homogeneity. *Games and Economic Behavior 54*(2), 241–260.

Curiel, M., M. Maschler, and S. Tijs (1987). Bankruptcy games. *Zeitschrift für Operations Research 31*.

Dagan, N. (1996). New characterizations of old bankruptcy rules. *Social Choice and Welfare 13*, 51–59.

Groenevelt, H. (1991). Two algorithms for maximizing a separable concave function over a polymatroid feasible region. *European Journal of Operational Research 54*(2), 227–236.

Harsanyi, J. (1959). A bargaining model for the cooperative N-person game. In R. Luce and A. Tucker (Eds.), *Contributions to the theory of games*, Volume IV. Princeton University Press.

Kalai, E. (1977). Proportional solutions to bargaining situations: Interpersonal utility comparisons. *Econometrica 45*(7), 1623–1630.

Kaminski, M. (2006). Parametric rationing methods. *Games and Economic Behavior 54*.

Moulin, H. (1987). Equal or proportional divsion of a surplus, and other methods. *International Journal of Game Theory 16*, 161–186.

Moulin, H. (2000). Priority rules and other asymmetric rationing methods. *Econometrica 68*(3), 643–684.

Moulin, H. (2002). Axiomatic cost and surplus-sharing. In K. Arrow, A. Sen, and K. Suzumura (Eds.), *Handbook of Social Choice and Welfare*, Volume 1, pp. 289–357. Elsevier.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences 2*, 345 – 371.

Schmeidler, D. (1979). A bibliographical note on a theorem by Hardy, Littlewood, and Polya. *Journal of Economic Theory 59*(20), 125–128.

Schummer, J. and W. Thomson (1997). Two derivations of uniform rule and an application to bankruptcy. *Economics Letters 55*(3), 333 – 337.

Stovall, J. (2014a). Asymmetric parametric division rules. *Games and Economic Behavior 84*(1), 87 – 110.

Stovall, J. (2014b). Collective rationality and monotone path division rules. *Journal of Economic Theory 154*, 1–24.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences 45*(3), 249 – 297.

Thomson, W. (2011). Consistency and its converse: an introduction. *Review of Economic Design 15*(4), 257 – 291.

Thomson, W. (2012). On the axiomatics of resource allocation: Interpreting the consistency principle. *Economics and Philosophy 28*(3), 385–421.

Thomson, W. (2014). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update.

Young, P. (1987). On dividing an amount according to claims or liabilities. *Mathematics of Operations Research 12*(3), 397 – 414.

Young, P. (1988). Distributive justice in taxation. *Journal of Economic Theory 44*, 321 – 335.

A Appendix

A.1 Proof of Lemma 4

As a preliminary step, we will also prove that the weighted priority solutions also satisfy the following property.

Resource monotonicity: For each $(c, e) \in \mathcal{P}^N$ and each e' such that $e \leq e' \leq \sum_N c_i$, $\varphi(c, e') \geq \varphi(c, e)$.

For each $i \in A$, let $(n_i, w_i) \in \mathbb{Z}_{++} \times \mathbb{R}_{++}$ be such that, if the resource comes in indivisible units, for each pair $i, j \in A$, $n_i \neq n_j$. For each $i \in A$ and $z_i \in \mathbb{R}_+$, let $f_i(z_i) \equiv n_i z_i + w_i \ln(1 + \frac{z_i}{w_i})$. Let φ denote the weighted priority solution defined as follows: for each $N \in \mathcal{N}$ and each $(c, e) \in \mathcal{P}^N$,

$$\varphi(c, e) \equiv \arg\max\{\textstyle\sum_{i \in N} f_i(z_i) : z \in Z(c, e)\}.$$

Let $N \in \mathcal{N}$. Note that, for each $(c, e) \in \mathcal{P}^N$, $a = \arg\max\{\sum_{i \in N} f_i(z_i) : z \in Z(c, e)\}$ if and only if,[14]

$$\text{for each pair } i, j \in N \text{ and } \varepsilon > 0, a + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in Z(c, e) \Rightarrow \partial_+ f_i(a_i) \leq \partial_- f_j(a_j)$$

where $\mathbf{e}_i \in \mathbb{R}^N_+$ is the standard basis vector with a 1 in the ith coordinate and zeros elsewhere, $\partial_+ f_i(a_i)$ is the right hand derivative of f_i at a_i and $\partial_+ f_j(a_j)$ is the left hand derivative of f_j at a_j.

Equivalently, for each $(c, e) \in \mathcal{P}^N$, $a = \arg\max\{\sum_{i \in N} f_i(z_i) : z \in Z(c, e)\}$ if and only if,

$$\text{for each pair } i, j \in N, [a_i < c_i, 0 < a_j] \Rightarrow \partial_+ f_i(a_i) \leq \partial_- f_j(a_j).$$

Equivalently,

$$\begin{aligned} &\text{for each } (c, e) \in \mathcal{P}^N,\ \ a = \arg\max\{\textstyle\sum_{i \in N} f_i(z_i) : z \in Z(c, e)\} \text{ if and only if,} \\ &\text{for each pair } i, j \in N, [a_i < c_i, 0 < a_j] \Rightarrow n_i + \frac{1}{1 + a_i/w_i} \leq n_j + \frac{1}{1 + a_j/w_j}. \end{aligned} \tag{10}$$

Let $(c, e) \in \mathcal{P}^N$ and let e' be such that $e < e' \leq \sum_{i \in N} c_i$. Let $x \equiv \varphi(c, e)$ and $x' \equiv \varphi(c, e')$.

Resource monotonicity: We will prove that $x' \geq x$. Otherwise, there is a pair $i, j \in N$ such that $x_i > x'_i$ and $x_j < x'_j$. Thus, $c_i \geq x_i > x'_i \geq 0$ and $0 \leq x_j < x'_j \leq c_j$. Thus, by (10),

$$n_i + \frac{1}{1 + x_i/w_i} \geq n_j + \frac{1}{1 + x_j/w_j} \quad \text{and} \quad n_i + \frac{1}{1 + x'_i/w_i} \leq n_j + \frac{1}{1 + x'_j/w_j}.$$

The first of these inequalities implies $n_i \geq n_j$ and the second one that $n_i \leq n_j$. Thus, $n_i = n_j$. Thus, $\frac{x_j}{w_j} \geq \frac{x_i}{w_i}$ and $\frac{x'_j}{w_j} \leq \frac{x'_i}{w_i}$. Thus, because by assumption $x_i > x'_i$ and $x_j < x'_j$, $\frac{x'_j}{w_j} > \frac{x_j}{w_j} \geq \frac{x_i}{w_i} > \frac{x'_i}{w_j}$, a contradiction. Thus, in fact, $x' \geq x$.

[14] This is a well known necessary and sufficient for the maximization of a separably concave function over a simple constraint set like ours. For instance, see Groenevelt (1991).

Composition up: We will prove that $x' = x + \varphi(c - x, e' - e)$ by establishing the first equality below

$$x' - x = \varphi(c - x, e' - e) = \arg\max\{\sum_{i \in N} f_i(z_i) : z \in Z(c - x, e' - e)\}$$

where the second equality follows from the definition of φ. Suppose this is not true and note that, as established above, $x' \geq x$. Then, by (10), there is a pair $i, j \in N$ such that

$$[x'_i - x_i < c_i - x_i, 0 < x'_j - x_j] \quad \text{and} \quad n_i + \frac{1}{1 + (x'_i - x_i)/w_i} > n_j + \frac{1}{1 + (x'_j - x_j)/w_j}.$$

Since $x' \geq x$, the second term in each side of the above inequality is no larger than one. Thus, since n_i and n_j are integers, $n_j \leq n_i$. Also note that

$$[x'_i - x_i < c_i - x_i, 0 < x'_j - x_j] \text{ implies } [x'_i < c_i \text{ and } 0 \leq x_j < x'_j].$$

Then, by (10),

$$n_i + \frac{1}{1 + x'_i/w_i} \leq n_j + \frac{1}{1 + x'_j/w_j}.$$

Thus, since n_i and n_j are integers and the second term in each side of the above inequality is no larger than one and $x'_j > 0$, $n_j \geq n_i$. Thus, $n_j = n_i$. Thus,

$$\frac{x'_j - x_j}{w_j} > \frac{x'_i - x_i}{w_i} \quad \text{and} \quad \frac{x'_j}{w_j} \leq \frac{x'_i}{w'_i}.$$

Thus,

$$\frac{x_i}{w_i} > \frac{x_j}{w_j}. \tag{11}$$

Thus, $x_i > 0$ and $x_j < x'_j \leq c_i$. Then, by (10) and $n_i = n_j$, $\frac{1}{1+x_i/w_i} \geq \frac{1}{1+x_j/w_j}$. Equivalently, $\frac{x_i}{w_i} \leq \frac{x_j}{w_j}$. This contradicts (11), establishing the desired conclusion.

Consistency: Let $N' \subseteq N$ and $y \equiv \varphi(c_{N'}, \sum_{N'} x_i)$. If $y \neq x_{N'}$, $y \in Z(c_{N'}, \sum_{N'} x_i)$ implies $\sum_{i \in N'} f_i(y_i) > \sum_{i \in N'} f_i(x_i)$. Then, $\sum_{i \in N'} f_i(y_i) + \sum_{i \in N \setminus N'} f_i(x_i) > \sum_{i \in N} f_i(x_i)$. Since $(y, x_{N \setminus N'}) \in Z(c, e)$, x does not maximize $\sum_{i \in N} f_i$ over $Z(c, e)$, a contradiction.

Converse consistency: Let $y \in Z(c,e)$ be such that, for each $\{i,j\} \subseteq N$, $y_{\{i,j\}} = \varphi(c_i, c_j, y_i + y_j)$. Since φ is *consistent*, $x_{\{i,j\}} = \varphi(c_i, c_j, x_i + x_j)$. Thus, by *consistency*, if there is $\{i,j\} \subseteq N$ such that $x_{\{i,j\}} \neq y_{\{i,j\}}$, $x_i + x_j \neq y_i + y_j$. If so, without loss of generality, $x_i + x_j > y_i + y_j$. By *resource monotonicity*,

$$x_{\{i,j\}} = \varphi((c_i, c_j), x_i + x_j) \geq \varphi((c_i, c_j), y_i + y_j) = y_{\{i,j\}}$$

and, without loss of generality, i is such that $x_i > y_i$. Thus, since $\sum_{i \in N} y_i = e = \sum_{i \in N} x_i$, there is $l \in N \setminus \{i,j\}$ such that $x_l < y_l$. By *consistency*, $x_{\{i,l\}} = \varphi(c_i, c_l, x_i + x_l)$ and, by assumption, $y_{\{i,l\}} = \varphi(c_i, c_k, y_i + y_l)$. Thus, if $x_i + x_l \geq y_i + y_l$, by *resource monotonicity*, $x_l \geq y_l$, which is not the case. Thus, $x_i + x_l < y_i + y_l$. Thus, by *resource monotonicity*, $x_{\{i,l\}} \leq y_{\{i,l\}}$, contradicting $x_i > y_i$. Thus, φ is *conversely consistent*.

Claims truncation invariance: Because $Z(c \wedge e, e) = Z(c,e)$, the maximizers of $\sum_{i \in N} f_i$ over $Z(c \wedge e, e)$ and $Z(c,e)$, respectively, coincide. Thus, $\varphi(c \wedge e, e) = \varphi(c,e)$.

A.2 Proof of Corollary 1

Suppose that the resource is divisible and let φ denote a solution satisfying *consistency*, *composition up*, *claims truncation invariance*, and *positive awards*. Then, by Theorem 1, for each $i \in A$, there is a positive integer n_i and a positive number w_i such that, for each $N \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^N$,

$$\varphi(c,e) = \arg\max\{\textstyle\sum_{i \in N} n_i x_i + w_i \ln(1 + x_i/w_i) : x \in Z(c,e)\}.$$

If *positive awards* were also imposed in the construction of the n_i and w_i parameters in Lemma 3, then, for each pair $i,j \in A$, $n_i = n_j$. Thus, there is an integer n_0 such that, for each $N \in \mathcal{N}$ and each $(c,e) \in \mathcal{P}^N$,

$$\varphi(c,e) = \arg\max\{\textstyle\sum_{i \in N} n_0 x_i + w_i \ln(1 + x_i/w_i) : x \in Z(c,e)\}.$$