Stability, Strategy-Proofness, and Cumulative Offer Mechanisms*

John William Hatfield
McCombs School of Business
University of Texas at Austin

Scott Duke Kominers
Society of Fellows
Harvard University

Alexander Westkamp
Department of Management,
Economics, and Social Sciences
University of Cologne

July 30, 2015

Abstract

We consider the setting of many-to-one matching with contracts, where firms may demand multiple contracts but each worker desires at most one contract. We introduce three novel conditions—observable substitutability, observable size monotonicity, and non-manipulatability—and show that when these conditions are satisfied, the cumulative offer mechanism is the unique mechanism that is stable and strategy-proof (for workers). Moreover, when the choice function of some firm fails any of our three conditions, one can construct unit-demand choice functions for the other firms such that no stable and strategy-proof mechanism exists. In the final part of the paper, we characterize the class of choice functions for which the cumulative offer mechanism is guaranteed to yield a stable outcome.

JEL Classification: C62; C78; D44; D47

Keywords: Matching with contracts, Stability, Strategy-proofness, Substitutability, Size monotonicity, Cumulative offer mechanism

*The authors appreciate the helpful comments of Fuhito Kojima and seminar audiences at MATCH-UP and the Society for Economic Design. Kominers gratefully acknowledges the support of National Science Foundation grants CCF-1216095 and SES-1459912 and the Harvard Milton Fund. Westkamp gratefully acknowledges funding from the People Programme (Marie Curie Intra-European Fellowship) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement 628276. Please e-mail comments or suggestions to john.hatfield@utexas.edu, kominers@fas.harvard.edu, and westkamp@wiso.uni-koeln.de.

1 Introduction

The development of stable and strategy-proof mechanisms for many-to-one matching with contracts has been a key focus of recent work in market design. In many-to-one matching with contracts, agents on one side of the market, e.g., workers, can fulfill at most one contract, while agents on the other side of the market, e.g., firms, may desire multiple contracts. Hatfield and Milgrom (2005) showed that when firms' preferences are *substitutable* and *size monotonic*, the worker-proposing cumulative offer mechanism is stable and strategy-proof for workers.[1,2] Recently, a number of authors have examined many-to-one matching with contracts settings where firms' preferences are not necessarily substitutable yet a stable and strategy-proof mechanism exists: Kamada and Kojima (2012, 2015) demonstrated the existence of a stable and strategy-proof mechanism for settings with regional caps in entry-level labor markets, focusing on medical-residency matching in Japan; Sönmez and Switzer (2013) and Sönmez (2013) showed that the cumulative offer mechanism is a stable and strategy-proof way to match cadets (at West Point and in the Reserve Officer Training Corps, respectively) to branches of service; Kominers and Sönmez (2015) suggested using the cumulative offer mechanism as a stable and strategy-proof system for allocating airline seat upgrades; and Dimakopoulos and Heller (2014) showed that the preferences of regional courts in Germany over lawyers seeking traineeships allow for stable and strategy-proof matching.

The numerous real-world applications of matching under non-substitutable preferences have motivated theoretical work to find weakened substitutability conditions on firms' preferences that still guarantee the existence of stable and strategy-proof mechanisms:

[1]Substitutability requires that whenever the set of contracts available to a hospital expands (in the superset sense), the set of contracts rejected by that hospital also expands. Size monotonicity requires that whenever the set of contracts available to a hospital expands (in the superset sense), the number of contracts chosen by the hospital weakly increases. Hatfield and Milgrom (2005) refer to size monotonicity as the "Law of Aggregate Demand."

[2]Henceforth, whenever we refer to a cumulative offer mechanism, we shall mean a cumulative offer mechanism in which agents on the side of the market with unit demand make proposals. Moreover, whenever we refer to strategy-proofness, we shall mean that a mechanism is strategy-proof for agents on the side of the market with unit demand. It is well-known that no stable mechanism can be strategy-proof for agents who can engage in more than one partnership (see Roth (1982)).

Hatfield and Kojima (2010) introduced *unilateral substitutability* and showed that when all firms' preferences are unilaterally substitutable (and size monotonic), the cumulative offer mechanism is stable and strategy-proof; Kominers and Sönmez (2015) provided a class of preferences, called *slot-specific*, and showed that if each firm's preferences are in this class, then the cumulative offer mechanism is stable and strategy-proof; and Hatfield and Kominers (2015) developed a concept of *substitutable completions* and showed that when each firm's preferences admit a size monotonic substitutable completion the cumulative offer mechanism is stable and strategy-proof.[3]

The earlier findings on weakened substitutability conditions for many-to-one matching with contracts are surprising, given that substitutability is sufficient and necessary for the *guaranteed existence* of stable outcomes in "pure" matching markets where the terms of each potential partnership are exogenously fixed (Hatfield and Kojima, 2008); that is, if each firm's preferences are substitutable, then a stable outcome always exists, and, if even one firm's preferences are not substitutable, there exist unit-demand preferences for the other firms and preferences for the workers such that no stable outcome exists.[4] Our paper is the first to develop a characterization of the sufficient and necessary conditions for the guaranteed existence of stable and strategy-proof mechanisms for many-to-one matching with contracts.[5]

Furthermore, our methods allow us to establish that whenever stable and strategy-proof mechanisms are guaranteed to exist, all such mechanisms are equivalent to the well-known cumulative offer mechanism. Our results can thus help explain the ubiquity of cumulative offer mechanisms: if all unit-demand hospital preferences are possible, then whenever stable and strategy-proof matching is feasible, the cumulative offer mechanism is the unique stable and strategy-proof mechanism. We also show that our sufficient conditions for the existence of

[3]Kadam (2015) showed that all unilaterally substitutable preferences are substitutably completable; Hatfield and Kominers (2015) showed that the slot-specific preferences of Kominers and Sönmez (2015) always admit a size monotonic substitutable completion.

[4]Furthermore, Hatfield and Kominers (2012, 2014) established that substitutability is necessary for the guaranteed existence of stable outcomes in many-to-many matching with contracts settings.

[5]In Section 2.4, we provide a formal discussion of what it means to "guarantee the existence of a stable and strategy-proof mechanism."

stable and strategy-proof mechanisms are strictly weaker than any previously known sufficient conditions.[6]

We consider the setting of many-to-one matching with contracts. In our model, each of a finite number of doctors desires to sign at most one contract with one of a finite number of hospitals.[7] There could be many (but a finite number of) different ways in which a given hospital can employ a given doctor.

We provide novel sufficient conditions on the preferences of hospitals such that a stable and strategy-proof mechanism exists. The key idea behind our conditions is that some violations of the substitutability and size monotonicity conditions are irrelevant for the existence of stable and strategy-proof mechanisms. For example, suppose that some hospital h always wants to employ doctor d, no matter which other contracts h has available or which contract with h doctor d has proposed. It might well be that the specific type of contract that h would like to sign with d depends on which other contracts h has available and hence the preferences of h would not be substitutable as the set of contracts rejected is non-monotonic in the set of contracts available to h. However, if we consider the doctor-proposing cumulative offer mechanism, this violation will never be observed since d will essentially dictate the terms of his employment with h by his first proposal to h. More generally, we say that a sequence of contracts with a given hospital h is *observable*, if, for each contract in the sequence, the doctor associated with that contract is not currently employed by h when h is allowed to choose from all previous contracts in the sequence. We say that the preferences of h are *observably substitutable* if the set of contracts rejected by h weakly expands along any observable sequence of contracts. Similarly, we say that the preferences of h are *observably size monotonic* if the number of contracts chosen by h weakly increases along any observable sequence of contracts. We show that, in contrast to the usual definitions of substitutability and size monotonicity, our concepts of observable substitutability and observable size monotonicity are necessary for

[6]Prior to our research, the completion-based conditions of Hatfield and Kominers (2015) were the weakest known sufficient conditions to guarantee the existence of a stable and strategy-proof mechanism.

[7]From now on, we use the terminology of doctors and hospitals instead of workers and firms to maintain consistency with the preceding literature on many-to-one matching with contracts.

the guaranteed existence of a stable and strategy-proof mechanism (Theorems 1 and 2).

Unfortunately, and somewhat surprisingly, observable substitutability and observable size monotonicity are *not* sufficient for the existence of a stable and strategy-proof mechanism. In particular, we present an example in which the cumulative offer mechanism always proceeds *as if* all hospitals had substitutable and size monotonic choice functions but is still manipulable by doctors. To complete our characterization, we introduce a third condition which requires that the choice function of hospital h is *non-manipulatable*, that is, if h is the only hospital, then the cumulative offer mechanism is strategy-proof. While the necessity of such a condition is straightforward (Theorem 3), it is far from obvious that such a condition will help us close the gap in our characterization. Nevertheless, our final main result (Theorem 4) shows that the combination of observable substitutability, observable size monotonicity, and non-manipulatability is sufficient for the existence of a stable and strategy-proof mechanism. Combining our results, we see that a stable and strategy-proof mechanism is guaranteed to exist if and only if hospitals' preferences are observably substitutable, observably size monotonic, and non-manipulatable.

Apart from characterizing the conditions under which a stable and strategy-proof mechanism is guaranteed to exist, it is also relevant, especially for practical purposes, to know more about the class of stable and strategy-proof mechanisms. We show that when the preferences of every hospital are observably substitutable, any stable and strategy-proof mechanism is equivalent to a cumulative offer mechanism (Proposition 2). Furthermore, cumulative offer mechanisms are order-independent, i.e., every cumulative offer mechanism produces the same outcome, regardless of the order in which proposals are made by doctors (Proposition 1). Hence, when a stable and strategy-proof mechanism is guaranteed to exist, it must be equivalent to *the* cumulative offer mechanism. This is an important extension of earlier uniqueness results in the literature. In contrast to earlier results, however, our uniqueness result is not a straightforward consequence of the structure of the set of stable outcomes.[8]

While observable substitutability is not enough to guarantee that the cumulative offer mechanism is stable and strategy-proof, it is enough to guarantee that the outcome of the cumulative offer mechanism is stable (Proposition 3). Moreover, when hospitals' preferences are observably substitutable, the cumulative offer mechanism is not manipulable via truncation strategies, i.e., a doctor can never obtain a strictly better outcome by simply increasing or decreasing the rank of the outside option (Proposition 4).

In the final part of our paper, we characterize the class of choice functions for which the cumulative offer mechanism is guaranteed to yield a stable outcome. We say that the preferences of a hospital h are *observably substitutable across doctors* if h never chooses a previously-rejected contract with a doctor not currently employed by h along any observable sequence of contracts. We show that, if the preferences of each hospital are observably substitutable across doctors, then the outcome of a cumulative offer mechanism is independent of proposal order (Proposition 5). Moreover, cumulative offer mechanisms are guaranteed to produce stable outcomes (Theorem 5). By contrast, if the preferences of any hospital are not observably substitutable across doctors, then there exist unit-demand preferences for the other hospitals such that no cumulative offer mechanism is stable (Theorem 6). However, we demonstrate by means of an example that there exists a larger class of firm preferences for which stable outcomes are guaranteed to exist. Hence, if one is only interested in achieving stable outcomes and does not care about incentive compatibility, it is not sufficient to restrict attention to cumulative offer mechanisms.

The remainder of the paper is organized as follows: Section 2 introduces the many-to-one matching with contracts framework. Section 3 proves our characterization results for stable and strategy-proof mechanisms. Section 4 provides conditions under which the cumulative

[8]In particular, our sufficient conditions for the existence of a stable and strategy-proof mechanism do *not* imply that there is a unique doctor-optimal stable outcome. Moreover, our proof technique is novel in that, unlike in the settings of Kominers and Sönmez (2015) and Hatfield and Kominers (2015), in our setting there is no natural way to construct an auxilary economy for which the existence of a doctor-optimal stable outcome is guaranteed.

offer process always produces a stable outcome. Section 5 concludes. Most of the proofs are presented in Appendix A.

2 Model

2.1 Framework

There is a finite set of *doctors* D and a finite set of *hospitals* H. There is also a finite set of *contracts* X, where each $x \in X$ is identified with a unique doctor $\mathsf{d}(x)$ and a unique hospital $\mathsf{h}(x)$; there may be many contracts between the same doctor-hospital pair. To simplify the statements of our results, we assume throughout that for each hospital h and each doctor d there exists at least one contract x such that $\mathsf{d}(x) = d$ and $\mathsf{h}(x) = h$. An *outcome* is a set of contracts $Y \subseteq X$. For an outcome Y, we let $\mathsf{d}(Y) \equiv \cup_{y \in Y}\{\mathsf{d}(y)\}$ and $\mathsf{h}(Y) \equiv \cup_{y \in Y}\{\mathsf{h}(y)\}$. For any $i \in D \cup H$, we let $Y_i \equiv \{y \in Y : i \in \{\mathsf{d}(y), \mathsf{h}(y)\}\}$. An outcome $Y \subseteq X$ is *feasible* if for all $d \in D$, $|Y_d| \leq 1$.

Each hospital $h \in H$ has multi-unit demand over contracts in X_h and is endowed with a choice function C^h that describes the hospital's choice from an available set of contracts, i.e., $C^h(Y) \subseteq Y$ for all $Y \subseteq X$. We assume throughout that for all $Y \subseteq X$ and all $h \in H$, hospital h

(1) only chooses contracts to which it is a party, i.e., $C^h(Y) \subseteq Y_h$,

(2) signs at most one contract with any given doctor, i.e., $C^h(Y)$ is feasible, and

(3) considers rejected contracts to be irrelevant, i.e., for all $x \in X$, if $x \notin C^h(\{x\} \cup Y)$, then $C^h(\{x\} \cup Y) = C^h(Y)$.[9]

A particularly simple class of choice functions for hospitals are the unit-demand choice functions; a hospital h has *unit demand* if $|C^h(Y)| \leq 1$ for all $Y \subseteq X$.

[9]The importance of this *irrelevance of rejected contracts* condition is discussed by Aygün and Sönmez (2012, 2013).

We denote by $C^H(Y) \equiv \cup_{h \in H} C^h(Y)$ the set of contracts chosen by the set of all hospitals from a set of contracts $Y \subseteq X$. For any $Y \subseteq X$ and $h \in H$, $R^h(Y) \equiv Y_h \smallsetminus C^h(Y)$ denotes the set of contracts that h rejects from Y. We denote by $R^H(Y) \equiv \cup_{h \in H} R^h(Y)$ the set of contracts rejected by the set of hospitals from a set of contracts $Y \subseteq X$.

Each doctor $d \in D$ has *unit demand* over contracts in X_d and an *outside option* $\emptyset$. We denote the strict preferences of doctor d over $X_d \cup \{\emptyset\}$ as $\succ_d$. A contract $x \in X_d$ is *acceptable* with respect to $\succ_d$ if $x \succ_d \emptyset$. We extend the specification of doctor preferences over contracts to preferences over outcomes in the natural way.[10]

2.2 Stability

We now define the concept of stability.

Definition 1. A feasible outcome $A \subseteq X$ is *stable* if it is

1. *Individually rational*: $C^H(A) = A$ and, for all $d \in D$, $A_d \succeq_d \varnothing$.

2. *Unblocked*: There does not exist a nonempty $Z \subseteq (X \smallsetminus A)$ such that $Z \subseteq C^H(A \cup Z)$ and, for all $d \in \mathsf{d}(Z)$, $Z \succ_d A$.

Our definition of stability is standard: we require that no agent wishes to unilaterally drop a contract, and that there does not exist a *blocking set* Z such that all hospitals and doctors associated with contracts in Z actually want to sign all contracts in Z—potentially after dropping some of the contracts in A.

[10] That is, for each doctor $d \in D$,

1. for any outcome Y such that $|Y_d| > 1$, we let $\varnothing \succ_d Y$,
2. for any outcome Y such that $Y_d = \varnothing$, we let $Y \sim_d \varnothing$,
3. for any two outcomes Y and Z such that $Y_d = \{y\}$ and $Z_d = \{z\}$, we let $Y \succ_d Z$ if and only if $y \succ_d z$,
4. for any two outcomes Y and Z such that $Y_d = \{y\}$ and $Z_d = \varnothing$, we let $Y \succ_d Z$ if and only if $y \succ_d \emptyset$, and,
5. for any two outcomes Y and Z such that $Y_d = \varnothing$ and $Z_d = \{z\}$, we let $Y \succ_d Z$ if and only if $\emptyset \succ_d z$.

2.3 Mechanisms

Given a profile of choice functions $C = (C^h)_{h \in H}$, a *mechanism* $\mathcal{M}(\cdot; C)$ maps preference profiles for the doctors $\succ = (\succ_d)_{d \in D}$ to outcomes. Most of the time, we shall assume that the choice functions of the hospitals are fixed and write $\mathcal{M}(\succ)$ in place of $\mathcal{M}(\succ; C)$. For future reference, we set $\mathcal{M}_d(\succ) \equiv [\mathcal{M}(\succ)]_d = \mathcal{M}(\succ) \cap X_d$ for all $d \in D$ and $\mathcal{M}_h(\succ) \equiv [\mathcal{M}(\succ)]_h = \mathcal{M}(\succ) \cap X_h$ for all $h \in H$.

A mechanism is *stable* if $\mathcal{M}(\succ)$ is a stable outcome for every preference profile $\succ$. A mechanism is *strategy-proof* if for every preference profile $\succ$, and for each doctor $d \in D$, there does not exist a $\hat{\succ}_d$ such that $\mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) \succ_d \mathcal{M}(\succ)$.

One class of mechanisms of particular importance is the class of *cumulative offer mechanisms.* A cumulative offer mechanism is defined with respect to a strict ordering $\vdash$ of the elements of X. For any preference profile $\succ$, the outcome of the cumulative offer mechanism, denoted by $\mathcal{C}^{\vdash}(\succ)$, is determined by the *cumulative offer process with respect to* $\vdash$ *and* $\succ$ as follows:

Step 0**:** Initialize the set of contracts *available* to the hospitals as $A^0 = \varnothing$.

Step $t \geq 1$**:** Consider the set

$$U^t \equiv \{x \in X : \mathsf{d}(x) \notin \mathsf{d}(C^H(A^{t-1})) \text{ and } \nexists z \in (X_{\mathsf{d}(x)} \smallsetminus A^{t-1}) \cup \{\emptyset\} \text{ such that } z \succ_{\mathsf{d}(x)} x\}.$$

If U^t is empty, then the algorithm terminates and the outcome is given by $C^H(A^t)$. Otherwise, letting y^t be the highest-ranked element of U^t according to $\vdash$, we set $A^t = A^{t-1} \cup \{y^t\}$ and proceed to step $t+1$.

A cumulative offer process begins with no contracts available to the hospitals (i.e., $A^0 = \varnothing$). Then, at each step t, we construct U^t, the set of acceptable contracts that both (1) have not yet proposed and (2) are not associated to doctors currently being chosen by hospitals. If U^t is empty, then every doctor either is chosen by some hospital or has no acceptable contracts left to propose, and so the cumulative offer process ends. Otherwise, the contract in U^t that is highest-ranked according to $\vdash$ is proposed by its associated doctor, and the process proceeds to the next step.

Letting T denote the last step of the cumulative offer process with respect to $\vdash$ and $\succ$, we call A^T the set of *observed* contracts in the cumulative offer process with respect to $\vdash$ and $\succ$. Note that without further assumptions on hospitals' choice functions, the outcome of a cumulative offer process need not be feasible, i.e., it might be the case that $C^H(A^T)$ contains more than one contract with a given doctor.

2.4 Guaranteeing Existence of Stable and Strategy-Proof Mechanisms

A *class* $\mathscr{C}^h$ for hospital h is a subset of the set of all possible choice functions for hospital h. We say that a class $\mathscr{C}^h$ is *unital* if it includes all unit-demand choice functions for h. A *profile of classes* $\mathscr{C} \equiv \times_{h \in H} \mathscr{C}^h$ is *unital* if, for each $h \in H$, $\mathscr{C}^h$ is unital.

A mechanism satisfying certain properties is *guaranteed to exist* for a profile of classes $\mathscr{C}$ if, whenever $C = (C^h)_{h \in H}$ is such that, for each $h \in H$, $C^h \in \mathscr{C}^h$, a mechanism $\mathcal{M}(\cdot; C)$ satisfying those properties exists.[11] Our main goal is to characterize the maximal unital profile of classes for which a stable and strategy-proof mechanism is guaranteed to exist.[12] That is, we wish to find the most general conditions on hospitals' choice functions that include every unit-demand choice function and, when imposed separately on the choice function of each hospital, guarantees the existence of a stable and strategy-proof mechanism.

The restriction to profiles of classes that contain unit-demand preferences is standard in matching theory. However, there are sets of profiles of choice functions, or domains, different

[11] For instance, Hatfield and Milgrom (2005) show that a stable and strategy-proof mechanism is guaranteed to exist for the profile of the classes of substitutable and size monotonic choice functions by showing that the cumulative offer mechanism for a given profile of substitutable and size monotonic choice functions is stable and strategy-proof.

[12] In particular, our results show that there is a unique profile of classes that assures existence and is maximal among all unital profiles of classes.

from the one that we will identify, for which a stable and strategy-proof mechanism exists. Our results show that such classes must either rule out some unit-demand choice functions or require some form of correlation across hospitals' preferences. We view the former restriction as problematic, given that unit demand is the most basic type of choice structure. Developing useful restrictions on the correlation across hospitals' preferences might be a more promising approach, although it is not clear how far one can go beyond trivial cases.[13]

3 Stable and Strategy-Proof Mechanisms

Most work on stable and strategy-proof matching mechanisms assumes the classical *substitutability* condition (see, e.g., Hatfield and Milgrom (2005)).[14] Our first condition weakens the substitutability condition by requiring the set of rejected contracts to expand only at sets of contracts that can be observed in cumulative offer processes. Consider an arbitrary hospital $h \in H$ whose choice function is given by C^h. An *offer process for* h is a finite sequence of distinct contracts $(x^1, \ldots, x^M)$ such that, for all $m = 1, \ldots, M$, $x^m \in X_h$. An offer process $(x^1, \ldots, x^M)$ for h is *observable* if, for all $m = 1, \ldots, M$, $\mathsf{d}(x^m) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{m-1}\}))$. Note that a set of contracts $Y \subseteq X_h$ can be observed in a cumulative offer process (where h is the only hospital) if and only if there is an observable offer process $(x^1, \ldots, x^M)$ such that $\{x^1, \ldots, x^M\} = Y$. Intuitively, this equivalence holds as a cumulative offer process only allows a doctor to make a new offer if that doctor does not have any contract currently held by some hospital. We can now present our first condition on on choice functions.

Definition 2. A choice function C^h *exhibits an observable violation of substitutability* if

[13] Another possible avenue is to develop preference restrictions that operate *across* the two market sides. In an important contribution, Pycia (2012) established a maximal domain result for the existence of stable outcomes in a class of coalition formation problems that includes many-to-one matching problems (without contracts). However, the characterization of Pycia implicitly relies on the existence of peer effects, that is, on the assumption that doctors care about more than just the hospitals they are assigned to. If there are no peer effects, the key preference restriction developed in Pycia (2012), *pairwise alignment*, is not necessary for the existence of stable outcomes, and also unlikely to be satisfied.

[14] The substitutability condition was introduced by Kelso and Crawford (1982) and adapted to settings with limited transferability by Roth (1984).

there exists an observable offer process $(x^1, \ldots, x^M) \in X_h$ such that $R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\}) \neq \varnothing$. A choice function C^h is *observably substitutable* if it does not exhibit an observable violation of substitutability.

Observable substitutability weakens classical substitutability by requiring rejected sets of contracts to expand only along observable offer processes, i.e., by requiring $R^h(Y) \subseteq R^h(Z)$ only if there exists an observable offer process $(x^1, \ldots, x^N)$ such that $\{x^1, \ldots, x^N\} = Z$ and, for some $M \leq N$, $\{x^1, \ldots, x^M\} = Y$. Note that if C^h exhibits an observable violation of substitutability, it is possible to detect a violation of substitutability by just observing proposed and (temporarily) chosen contracts in a cumulative offer process. It is intuitive that if some choice function C^h exhibits an observable violation of substitutability, then some cumulative offer mechanism may fail to be stable and strategy-proof. In fact, our first result shows that, for any unital profile of classes, observable substitutability is necessary to guarantee the existence of *any* stable and strategy-proof mechanism.

Theorem 1. *Suppose that $|H| > 1$ and that the choice function of some hospital is not observably substitutable. Then there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.*

Before introducing the second condition for our characterization, we derive four important implications of observable substitutability. First, we show that observable substitutability is sufficient for the outcome of a cumulative offer process to be independent of the order of proposals.

Proposition 1. *Suppose that the choice function of every hospital is observably substitutable. For any preference profile $\succ$ and any two orderings $\vdash, \vdash'$, the set of all contracts available to hospitals at the end of the the cumulative offer process for $\vdash$ coincides with the set of all contracts available to hospitals at the end of the cumulative offer process for $\vdash'$.*[15]

[15]This result follows immediately from Proposition 5.

In light of Proposition 1, we can, for any fixed profile of observably substitutable choice functions, define *the* cumulative offer mechanism as a mapping $\mathcal{C}$ from preference profiles of doctors into outcomes. Second, we show that for any profile of observably substitutable choice functions, the cumulative offer mechanism is essentially the only candidate for a stable and strategy-proof mechanism.

Proposition 2. *Suppose that the choice function of every hospital is observably substitutable. If $\mathcal{M}$ is a stable and strategy-proof mechanism, then, for any preference profile $\succ$, $\mathcal{M}(\succ) = \mathcal{C}(\succ)$.*

The third implication of observable substitutability is the guaranteed stability of the cumulative offer mechanism.

Proposition 3. *Suppose that the choice function of every hospital is observably substitutable. For any preference profile $\succ$, $\mathcal{C}(\succ)$ is stable.*[16]

Finally, we will show that observable substitutability is sufficient to rule out the possibility that doctors can manipulate the cumulative offer mechanism with a particularly simple type of strategy: Given a doctor $d \in D$ and a preference relation $\succ_d$, say that $\hat{\succ}_d$ is a *truncation of* $\succ_d$, if, for all $x, y \in X_d$, $x \succ_d y$ if and only if $x \mathrel{\hat{\succ}_d} y$. That is, a truncation strategy is a strategy where a doctor keeps the same ranking over contracts, but changes which contracts he finds acceptable. We show that observable substitutability is sufficient for truthtelling to be weakly dominant in the space of truncation strategies for the cumulative offer mechanism.

Proposition 4. *Suppose that the choice function of every hospital is observably substitutable. Then for all preference profiles $\succ$, all doctors $d \in D$, and all truncations $\hat{\succ}_d$ of $\succ_d$, we have that $\mathcal{C}(\succ) \succeq_d \mathcal{C}(\hat{\succ}_d, \succ_{-d})$.*

Together, Propositions 3 and 4 show that, when doctors can only use truncation strategies, the cumulative offer mechanism will produce a stable outcome and doctors will truthfully report their preferences.

[16] This result follows immediately from Theorem 5.

However, as is to be expected given the results in the prior literature, observable substitutability by itself is not sufficient to rule out profitable manipulations that do not take the form of a truncation. We now introduce a weakening of the Law of Aggregate Demand/size monotonicity condition of Hatfield and Milgrom (2005). This condition will play a crucial role in our characterization result.

Definition 3. A choice function C^h *exhibits an observable violation of size monotonicity* if there exists an offer process $(x^1, \ldots, x^M) \in X_h$ such that $|C^h(\{x^1, \ldots, x^M\})| < |C^h(\{x^1, \ldots, x^{M-1}\})|$. A choice function C^h is *observably size monotonic* if it does not exhibit an observable violation of size monotonicity.

Our next result shows that, for any unital profile of classes, observable size monotonicity is necessary to guarantee the existence of a stable and strategy-proof mechanism.

Theorem 2. *Suppose that $|H| > 1$ and that the choice function of some hospital is observably substitutable but not observably size monotonic. Then there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.*

Theorems 1 and 2 show that observable substitutability and observable size monotonicity are necessary for the existence of a stable and strategy-proof mechanism. Unfortunately, observable substitutability and observable size monotonicity are not sufficient for the cumulative offer mechanism (or any other mechanism) to be stable and strategy-proof.

Example 1. Consider a setting in which $H = \{h\}, D = \{d, e\}$, and $X = \{x, \hat{x}, y, \hat{y}\}$, with $\mathsf{h}(x) = \mathsf{h}(\hat{x}) = \mathsf{h}(y) = \mathsf{h}(\hat{y}) = h$, $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$ and $\mathsf{d}(y) = \mathsf{d}(\hat{y}) = e$. Let the choice function C^h of h be induced by the preferences

$$\{\hat{y}\} \succ \{x, y\} \succ \{x\} \succ \{\hat{x}\} \succ \{y\} \succ \varnothing.$$

The choice function C^h is observably substitutable and observably size monotonic.

If the preferences of the doctors are given by

$$d : \hat{x} \succ x \succ \emptyset$$
$$e : y \succ \hat{y} \succ \emptyset,$$

then the cumulative offer process produces the outcome $\{\hat{y}\}$. However, if $d = \mathsf{d}(x)$ reports his preferences as $x \succ \emptyset$, the cumulative offer process produces the outcome $\{x, y\}$, under which d is strictly better off. Hence, the cumulative offer mechanism is not strategy-proof. Therefore, by Proposition 2, we see that no stable and strategy-proof mechanism exists.

Example 1 is somewhat surprising since it shows that even when the cumulative offer mechanism runs *as if* hospitals had substitutable and size monotonic preferences, a stable and strategy-proof mechanism may fail to exist. Our third and final condition rules out situations such as the one encountered in the previous example.

Definition 4. The choice function C^h of hospital h is *manipulatable* if it is observably substitutable but there exists a preference profile $\succ$ that only ranks contracts with h as acceptable, a doctor $d \in D$, and a preference relation $\hat{\succ}_d$ that only ranks contracts with h as acceptable such that $\mathcal{C}(\hat{\succ}_d, \succ_{D \setminus \{d\}}) \succ_d \mathcal{C}(\succ)$.

Note that, by Proposition 2, when choice functions are observably substitutable, any stable and strategy-proof mechanism has to coincide with the cumulative offer mechanism. Hence, the non-manipulatability of the choice function of h essentially requires that the only candidate for a stable and strategy-proof mechanism, the cumulative offer mechanism, is strategy-proof in a fictitious economy where h is the *only* available employer. The necessity of such a condition is almost tautological.

Theorem 3. *Suppose that the choice function C^h of hospital h is manipulatable. Then no stable and strategy-proof mechanism exists.*

Proof. By assumption, there exists a preference profile $\succ$ that only ranks contracts with

h as acceptable, a doctor $d \in D$, and a preference relation $\hat{\succ}_d$ that only ranks contracts with h as acceptable such that $\mathcal{C}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) \succ_d \mathcal{C}(\succ)$. Since C^h is observably substitutable, Proposition 2 implies that for any stable and strategy-proof mechanism $\mathcal{M}$, we have $\mathcal{M}(\succ) = \mathcal{C}(\succ)$ and $\mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) = \mathcal{C}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}})$. Hence, $\mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) \succ_d \mathcal{M}(\succ)$, contradicting the strategy-proofness of $\mathcal{M}$. $\square$

Since h was the only hospital in Example 1, the choice function of h in that example was manipulatable given that doctor $e = \mathsf{d}(y)$ was able to profitably manipulate the cumulative offer mechanism. Our final result, Theorem 4, shows that when the choice function of each hospital is observably substitutable, observably size monotonic, and non-manipulatable, then the cumulative offer process is strategy-proof.

Theorem 4. *Suppose that the choice function of every hospital is observably substitutable, observably size monotonic, and non-manipulatable. Then the cumulative offer mechanism is stable and strategy-proof.*

Before discussing the proof strategy, we discuss the practical relevance of our results. First, our conditions for the existence of a stable and strategy-proof mechanism can be checked independently at each hospital, and do not depend on subtle interactions between hospitals' choice functions. Second, while observable substitutability, observable size monotonicity, and non-manipulatability are not easy to verify, it is not clear that it is significantly more demanding to verify than the standard substitutability or size monotonicity conditions.[17] Third, and most importantly, our characterization result establishes that the cumulative offer mechanism is stable and strategy-proof whenever the very existence of a stable and strategy-proof mechanism can be guaranteed. This provides an important justification for the use of the cumulative offer mechanism when only limited information about hospitals' preferences is available.

[17] Hatfield et al. (2012) show that a substitutability checking algorithm that only has access to choice functions has a running time that is exponential in the number of available contracts.

Moreover, our set of conditions allows for choice functions under which the existence of a stable and strategy-proof mechanism could not be heretofore guaranteed. Hatfield and Kominers (2015) provided the most general sufficient conditions for the guaranteed existence of stable and strategy-proof mechanisms that were known prior to our work. Specifically, Hatfield and Kominers showed that when each hospital's choice function has a substitutable and size monotonic *completion*, the cumulative offer mechanism is stable and strategy-proof; a *completion* of a choice function C^h of hospital $h \in H$ is a choice function $\bar{C}^h$ such that for all $Y \subseteq X$, either

- $\bar{C}^h(Y) = C^h(Y)$, or
- there exist distinct $z, \hat{z} \in \bar{C}^h(Y)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$.

Our Example 2 provides an example of a choice function that is observably substitutable, observably size monotonic, and non-manipulatable—and yet does not have a substitutable completion.

Example 2. Consider a setting in which $H = \{h\}, D = \{d, e, f\}$, and $X = \{x, y, z, \hat{x}, \hat{z}\}$, with $\mathsf{h}(x) = \mathsf{h}(y) = \mathsf{h}(z) = \mathsf{h}(\hat{x}) = \mathsf{h}(\hat{z}) = h$, $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$, $\mathsf{d}(y) = e$, and $\mathsf{d}(z) = \mathsf{d}(\hat{z}) = f$. Let the choice function C^h of h be induced by the preferences

$$\{\hat{x}, z\} \succ \{\hat{z}, x\} \succ \{\hat{z}, y\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{z, y\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \succ$$
$$\succ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

It is straightforward to check that C^h is observably substitutable, observably size monotonic, and non-manipulatable.[18]

[18] In order to see that C^h is non-manipulatable, note first that x can never be proposed and rejected in the cumulative offer mechanism: If $\hat{z}$ is proposed, the cumulative offer mechanism will choose the outcome $\{\hat{z}, x\}$; if $\hat{z}$ is not proposed and y is proposed, the cumulative offer mechanism will choose the outcome $\{x, y\}$; if $\hat{z}$ and y are both not proposed, the cumulative offer mechanism will choose $\{x, z\}$ if z is proposed, and $\{x\}$ if z is not proposed. Given that x can not be proposed and rejected in the cumulative offer mechanism, it is easy to see that d can not profitably manipulate the cumulative offer mechanism with h as the only available hospital. Similar arguments show that $\hat{z}$ can not be proposed and rejected in the cumulative offer mechanism.

However, C^h does not have a substitutable completion. To see this, suppose that a substitutable completion $\bar{C}^h$ exists (with an accompanying rejection function $\bar{R}^h$). By the definition of completion, $C^h(Y) = \bar{C}^h(Y)$ for all $Y \subseteq X$ such that $|\mathsf{d}(Y)| = |Y|$, i.e., for all $Y \subseteq X$ that contain at most one contract with each doctor; hence $R^h(Y) = \bar{R}^h(Y)$ for all such Y. Hence,

$$\hat{x} \in R^h(\{\hat{x}, \hat{y}, \hat{z}\}) \Rightarrow \hat{x} \in \bar{R}^h(\{\hat{x}, \hat{y}, \hat{z}\})$$
$$z \in R^h(\{x, y, z\}) \Rightarrow z \in \bar{R}^h(\{x, y, z\})$$
$$y \in R^h(\{\hat{x}, y, z\}) \Rightarrow y \in \bar{R}^h(\{\hat{x}, y, z\}),$$

as each set of contracts considered contains at most one contract with each doctor. Combining these three facts about $\bar{R}^h$, we have that $\bar{C}^h(X) \subseteq \{\hat{z}, x\}$ as $\bar{C}^h$ is substitutable. But then $\bar{C}^h(X) = C^h(X)$, as $\bar{C}^h$ is a completion of C^h; but $C^h(X) = \{\hat{x}, z\} \nsubseteq \{\hat{z}, x\}$, a contradiction.

The proof of Theorem 4 starts from the assumption that, at some preference profile $\succ$, some doctor $\hat{d}$ can profitably manipulate the cumulative offer process by submitting $\hat{\succ}_{\hat{d}}$ instead of $\succ_{\hat{d}}$ since the former yields a strictly more preferred contract $\hat{x}$. In our proof, we establish that the choice function of $\hat{h} \equiv \mathsf{h}(\hat{x})$ must be manipulatable. The idea is to remove all contracts with hospitals other than $\hat{h}$ from $\succ$ and $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{-\hat{d}})$, leading to the preference profiles $\succ'$ and $\hat{\succ}'$. Intuitively, this deletion of contracts with other hospitals increases the competition for contracts with the one remaining hospital $\hat{h}$. Since $\hat{d}$ preferred $\hat{x}$ to the contract that he obtains under the cumulative offer process under $\succ$, $\hat{d}$ must prefer $\hat{x}$ to the contract, if any, that he obtains under the cumulative offer process under $\succ'$. The more difficult part of the proof is to show that the increased competition for contracts with $\hat{h}$ at $\hat{\succ}'$ does not hurt $\hat{d}$ in the sense that $\hat{x}$ is not rejected during the cumulative offer process for $\hat{\succ}'$. The next example illustrates the ideas that we use in the proof.

Hence, f can also not profitably manipulate the cumulative offer mechanism with h as the only available hospital. It is clear that e cannot profitably manipulate the cumulative offer mechanism since there is just one contract associated with e.

Example 3. Consider a setting in which $H = \{\hat{h}, h'\}, D = \{\hat{d}, e, f\}$, and

$$X = \{x, \hat{x}, y, \tilde{y}, y', w, w', w''\},$$

with $\mathsf{h}(x) = \mathsf{h}(\hat{x}) = \mathsf{h}(y) = \mathsf{h}(\tilde{y}) = \mathsf{h}(w) = \hat{h}$, $\mathsf{h}(y') = \mathsf{h}(w') = \mathsf{h}(w'') = h'$, $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = \hat{d}$, $\mathsf{d}(y) = \mathsf{d}(\tilde{y}) = e$, and $\mathsf{d}(w) = \mathsf{d}(w') = \mathsf{d}(w'') = f$.

We assume that the true preferences of the doctors are

$$\begin{aligned}
&\hat{d} : x \succ \hat{x} \succ \emptyset \\
&e : y \succ y' \succ \tilde{y} \succ \emptyset \\
&f : w' \succ w \succ w'' \succ \emptyset.
\end{aligned}$$

Now assume that $C^{h'}$ and $C^{\hat{h}}$ are both observably substitutable and observably size monotonic. Assume also that $C^{h'}$ and C^{h} are such that $[\mathcal{C}(\succ)]_{\hat{d}} = \varnothing$ and that, for $\hat{\succ}_{\hat{d}}$: $\hat{x} \succ \emptyset$, we have $\hat{x} \in \mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{-\hat{d}})$ so that $\hat{d}$ can profitably manipulate at $\succ$. Consider the cumulative offer mechanism with the ordering

$$y \vdash \tilde{y} \vdash y' \vdash w \vdash w' \vdash w'' \vdash x \vdash \hat{x}.$$

One possibility is the following:[19]

- The cumulative offer process under $\succ$ produces the offer process $(y, w', x, y', w, w'', \tilde{y}, \hat{x})$ and $\hat{x} \in R^{\hat{h}}(\{y, w', x, y', w, w'', \tilde{y}, \hat{x}\}) \smallsetminus R^{\hat{h}}(\{y, w', x, y', w, w'', \tilde{y}\})$, and
- The cumulative offer process under $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{-\hat{d}})$ produces the offer process $(y, w', \hat{x})$ and $\hat{x} \in C^{\hat{h}}(\{y, w', \hat{x}\})$.

Now consider the preferences $\succ'$, which are constructed by declaring all contracts with h' unacceptable while maintaining the ordering of contracts with h specified by $\succ$. It is easy to

[19] One possible preference ordering for hospital $\hat{h}$ that is compatible with everything that follows is the ordering from Example 1: $\{\tilde{y}\} \succ \{\hat{x}, y\} \succ \{\hat{x}\} \succ \{x\} \succ \{y\}$.

see that all contracts in $X_{\hat{h}}$ have to be proposed in the cumulative offer process with respect to $\succ'$ and $\vdash$. In particular, $\hat{x}$ must be rejected in the cumulative offer process with respect to $\succ'$ and $\vdash$.

Next, consider the preferences $\hat{\succ}'$, which are constructed by declaring all contracts with h' unacceptable while maintaining the same ordering for all contracts with h as in $\hat{\succ}$. We will now show that the increased competition for contracts with $\hat{h}$ at $\hat{\succ}'$ that results from removing contracts with h' from $\hat{\succ}$ will not hurt doctor $\hat{d}$. Note $\hat{h}$ must receive at least one more proposal, contract w, due to the increased competition. We will now outline key arguments from our proof of Theorem 4 by showing that $\hat{x} \notin R^h(\{y, \hat{x}, w\})$.

Note first that we must have $y \in C^h(\{y\})$. This follows since $\vdash$ ranks all of the contracts associated with doctor e before the contracts with other doctors and since w' is the second contract proposed in the cumulative offer process with respect to $\succ$ and $\vdash$. Next, note that observable size monotonicity implies $|C^h(\{y, w\})| = 1$ since the rules of the cumulative offer process with respect to $\succ$ and $\vdash$ require $C^h(\{x, y, w\}) = \{x\}$. Hence, we must have $R^h(\{y, w\}) \neq \varnothing$. Now consider the cumulative offer process with respect to $\hat{\succ}$ and $\vdash$. Since $\hat{d}$ and e each only propose one contract, we must have $C^h(\{\hat{x}, y\}) = \{\hat{x}, y\}$. Observable size monotonicity then implies that $|C^h(\{\hat{x}, y, w\}| \geq 2$. On the other hand, observable substitutability implies $R^h(\{y, w\}) \subseteq R^h(\{\hat{x}, y, w\})$. Given that $R^h(\{\hat{x}, y\}) = \varnothing$, we obtain $R^h(\{y, w\}) \subseteq R^h(\{\hat{x}, y, w\}) \smallsetminus R^h(\{\hat{x}, y\})$. Observable size monotonicity then implies $R^h(\{y, w\}) = R^h(\{\hat{x}, y, w\}) \smallsetminus R^h(\{\hat{x}, y\})$. Hence, $\hat{x} \notin R^h(\{\hat{x}, y, w\})$ since, obviously, $\hat{x} \notin R^h(\{y, w\})$.

The proof strategy for establishing Theorem 4 is novel; in particular, the argument here differs from that used by Hatfield and Milgrom (2005) and Hatfield and Kojima (2010) to prove that the cumulative offer mechanism is strategy-proof in their settings. Hatfield and Milgrom (2005) show that when hospitals' preferences are substitutable there exists a doctor-optimal stable outcome, i.e., a stable outcome weakly preferred by every doctor to every other stable outcome; moreover, when the hospitals' preferences are, in addition, size monotonic, the same set of doctors is employed at every stable outcome (a result known as the rural hospitals theorem). These results together imply that a mechanism which always selects the doctor-optimal stable outcome, such as the cumulative offer mechanism, is strategy-proof for doctors. Hatfield and Kojima (2010) showed analogous results while requiring only that hospitals' preferences are unilaterally substitutable. But as Example 4 demonstrates below, even when the preferences of each hospital are observably substitutable, observably size monotonic, and non-manipulatable, there does not necessarily exist a doctor-optimal stable outcome.

Example 4. Consider the setting of Example 2 and let, as in Example 2, the choice function of h be induced by

$$\{\hat{x}, z\} \succ \{\hat{z}, x\} \succ \{\hat{z}, y\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{z, y\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \succ$$
$$\succ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

Let the preferences of the doctors be given by

$$d : x \succ \hat{x} \succ \emptyset$$
$$e : y \succ \emptyset$$
$$f : z \succ \hat{z} \succ \emptyset.$$

There are two stable outcomes: $\{\hat{x}, z\}$ and $\{\hat{z}, x\}$, with the former preferred by $f = \mathsf{d}(z)$ and the latter preferred by $d = \mathsf{d}(x)$. Nevertheless, any cumulative offer process produces the same stable outcome, $\{\hat{z}, x\}$, and the cumulative offer mechanism is strategy-proof.

We note also that in the setting of Example 4, we can not use the techniques of Kominers and Sönmez (2015) or Hatfield and Kominers (2015) to construct an auxiliary economy in which a doctor-optimal stable outcome exists since (as demonstrated in Example 2) the choice function of hospital h in Example 4 is not substitutably completable (and, thus, in particular, can not be represented by slot-specific preferences).

We end this section by summarizing our results in the following corollary.

Corollary 1. *Let $\mathscr{C}$ be a unital profile of classes and suppose that $|H| > 1$. The following are equivalent:*

(i) For all $h \in H$, and for all $C^h \in \mathscr{C}^h$, C^h is observably substitutable, observably size monotonic, and non-manipulatable.

(ii) Any cumulative offer mechanism is stable and strategy-proof for $\mathscr{C}$.

(iii) A stable and strategy-proof mechanism is guaranteed to exist for $\mathscr{C}$.

Furthermore, if for each $C \in \mathscr{C}$ a stable and strategy-proof mechanism $\mathcal{M}$ exists, then for each $C \in \mathscr{C}$ all cumulative offer mechanisms are equivalent and $\mathcal{M} = \mathcal{C}$.

In independent work, Hirata and Kasuya (2015) have shown that there exists at most one stable and strategy-proof mechanism for any profile of choice functions that satisfies the irrelevance of rejected contracts condition. Hirata and Kasuya do not provide any characterization of conditions under which a stable and strategy-proof mechanism is guaranteed to exist, nor do they characterize the class of mechanisms that *could* be stable and strategy-proof. However, Hirata and Kasuya do establish uniqueness of the stable and strategy-proof mechanism for any given profile of choice functions. By contrast, our methods allow us to establish that there is at most one stable and strategy-proof mechanism for any profile of observably substitutable choice functions, and that, if even one hospital's choice function is not observably substitutable, there exist unit-demand choice functions for the other hospitals such that no stable and strategy-proof mechanism exists.

Finally, by Corollary 1, we see that our three conditions subsume all previously known sufficient conditions for the existence of stable and strategy-proof mechanisms. In particular, any choice function that either

1. is unilaterally substitutable and size monotonic,

2. is induced by slot-specific priorities, or

3. has a substitutable and size monotonic completion

must be observably substitutable, observably size monotonic, and non-manipulatable. By contrast, Example 2 shows that the combination of observably substitutability, observably size monotonicity, and non-manipulatability is strictly weaker than any of the previously known sets of conditions guaranteeing the existence of a stable and strategy-proof mechanism.

4 Stable Outcomes and Cumulative Offer Mechanisms

The results of Section 3 show that when one is interested in the existence of a stable and strategy-proof mechanism, attention can be restricted to the cumulative offer mechanism. This begs the question of whether the restriction to the cumulative offer mechanism is also without loss of generality when the only constraint is that a stable outcome is to be reached. To answer this question, we first introduce a weakening of the observable substitutability condition.

Definition 5. A choice function C^h is *observably substitutable across doctors*, if, for any observable offer process $(x^1, \ldots, x^M) \in X_h$, we have that if $x \in R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\})$ then $\mathsf{d}(x) \in \mathsf{d}(C^h(\{x^1, \ldots, x^{M-1}\}))$.

Note that observable substitutability across doctors is weaker than observable substitutability given that the latter requires $R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\}) = \varnothing$ for any observable offer process $(x^1, \ldots, x^M)$. By contrast, observable substitutability across doctors requires that whenever a hospital chooses a previously-rejected contract x during an observable offer process, the hospital previously chose some contract x' with the same doctor, i.e., $\mathsf{d}(x) = \mathsf{d}(x')$.[20]

[20] Hatfield and Kojima (2010) refer to this as "renegotiation," as the hospital and doctor "renegotiate" the terms of the doctor's employment to their mutual benefit. Such renegotiation does not take place during a cumulative offer process if the choice function of a hospital is observably substitutable.

The first result of this section is that observable substitutability across doctors is sufficient for cumulative offer processes to be independent of the order of proposals.

Proposition 5. *Suppose that the choice function of every hospital is observably substitutable across doctors. For any preference profile $\succ$ and any two orderings $\vdash, \vdash'$, the set of all contracts available to hospitals at the end of the the cumulative offer process for $\vdash$ coincides with the set of all contracts available to hospitals at the end of the cumulative offer process for $\vdash'$.*[21]

Our second result shows that observable substitutability across doctors implies that *the* cumulative offer mechanism always produces a stable outcome.

Theorem 5. *If the choice function of every hospital is observably substitutable across doctors, then the cumulative offer mechanism is stable.*

Before discussing the necessity of observable substitutability across doctors for the cumulative offer mechanism to produce stable outcomes, we discuss the relationship of observable substitutability across doctors with bilateral substitutability, a weakening of the classic substitutability condition introduced by Hatfield and Kojima (2010). A choice function C^h is *bilaterally substitutable*, if for every set of contracts $Y \subseteq X$, and every pair of contracts $x, z \in X \smallsetminus Y$ such that $\mathsf{d}(x), \mathsf{d}(z) \notin \mathsf{d}(Y)$, $z \notin C^h(Y \cup \{z\})$ implies that $z \notin C^h(Y \cup \{x, z\})$. Hatfield and Kojima (2010) showed that bilateral substitutability of hospitals' choice functions is sufficient to ensure that, for any preference profile of the doctors and any ordering of contracts, the corresponding cumulative offer mechanism yields a stable outcome (Theorem 1 of Hatfield and Kojima, 2010). It is straightforward to show that the bilateral substitutability condition implies observable substitutability across doctors and that observable substitutabil-

[21] Prior to our work, Hirata and Kasuya (2014) showed that cumulative offer processes are order-independent when each firm's preferences are bilaterally substitutable, and Hatfield and Kominers (2015) showed a similar result when when each firm's preferences are substitutably completable. In Appendix B.2, we provide an example of observably substitutable preferences that are neither bilaterally substitutable nor substitutably completable.

ity across doctors is strictly weaker than bilateral substitutability.[22,23] Hence, Theorem 5 implies Theorem 1 of Hatfield and Kojima (2010).

The final result of this section shows that, for any unital profile of classes, observable substitutability across doctors is necessary to guarantee the stability of cumulative offer mechanisms.

Theorem 6. *Suppose that $|H| > 1$ and that the choice function of some hospital is not observably substitutable across doctors. Then there exist unit-demand choice functions for the other hospitals such that no cumulative offer mechanism is stable.*

Before proceeding, we discuss the relationship between our results on the stability of cumulative offer mechanisms and the work of Flanagan (2014). Flanagan defines a condition called *cumulative offer revealed bilateral substitutability* and argues, somewhat informally, that this condition is sufficient for the cumulative offer mechanism to produce a stable outcome.[24] While it is clear that any choice function that satisfies the cumulative offer revealed bilateral substitutability condition is observably substitutable across doctors, it is an open question whether there exists a choice function that is observably substitutable across doctors but does not satisfy the cumulative offer revealed bilateral substitutability condition. Our contributions in Section 4 relative to those of Flanagan (2014) are that we

[22]Suppose that C^h is not observably substitutable across doctors. Let $(x^1, \ldots, x^M) \in X_h$ be an observable offer process and $x \in \{x^1, \ldots, x^M\}$ be a contract such that $x \in R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\})$ even though $\mathsf{d}(x) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{M-1}\}))$. Set $Y \equiv C^h(\{x^1, \ldots, x^{M-1}\}) \cup (C^h(\{x^1, \ldots, x^M\}) \smallsetminus \{x, x^M\})$ and note that irrelevance of rejected contracts implies $C^h(Y \cup \{x\}) = C^h(\{x^1, \ldots, x^{M-1}\})$ and $C^h(Y \cup \{x, x^M\}) = C^h(\{x^1, \ldots, x^M\})$. Since $(x^1, \ldots, x^M)$ is observable, $\mathsf{d}(x^M) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{M-1}\}))$. By the construction of Y, this implies $\mathsf{d}(x), \mathsf{d}(x^M) \notin \mathsf{d}(Y)$. This shows that C^h is not bilaterally substitutable.

[23]In their Appendix D, Hatfield and Kominers (2015) provided an example of a choice function that is observably substitutable across doctors (and, in fact, substitutably completable) but not bilaterally substitutable. Our Appendix B.2 also presents an example of an observably substitutable choice function which is not bilaterally substitutable.

[24]Flanagan (2014) verbally defines cumulative offer revealed bilateral substitutability as follows: "For any market and any execution of the cumulative offer process, I say that f reveals preferences during the cumulative offer process consistent with [bilateral substitutes] if there exists a preference [relation for f that satisfies bilateral substitutes and] which would generate an identical procedure. Contracts are cumulative offer revealed bilateral substitutes for f, if, for every [preference profile of workers and firms, such that all other firms' preferences satisfy bilateral substitutes], the preferences revealed by f during the cumulative offer process are consistent with [bilateral substitutes]" (p. 115, Flanagan (2014)).

1. show that observable substitutability across doctors is sufficient to guarantee that cumulative offer processes are independent of the order of proposals,
2. provide a full formal proof that observable substitutability across doctors is sufficient for the cumulative offer mechanism to be stable, and
3. establish that *no* cumulative offer mechanism can be guaranteed to yield stable outcomes when observable substitutability across doctors is violated and all unit-demand choice functions are allowed.

Unfortunately, observable substitutability across doctors is not necessary and sufficient for the existence of stable outcomes—our next example shows that it is not the case that if the choice function of some hospital is not observably substitutable across doctors, then there necessarily exist unit-demand choice functions for the other hospitals and preferences for the doctors such that no stable outcome exists.

Example 5. Consider the setting in which $H = \{h\}$, $D = \{d, e, f, g\}$, and $X = \{w, x, \hat{x}, y, z, \hat{z}\}$, with $\mathsf{h}(w) = \mathsf{h}(x) = \mathsf{h}(\hat{x}) = \mathsf{h}(y) = \mathsf{h}(z) = \mathsf{h}(\hat{z}) = h$, $\mathsf{d}(x) = \mathsf{d}(\hat{x}) = d$, $\mathsf{d}(y) = e$, $\mathsf{d}(z) = \mathsf{d}(\hat{z}) = f$, and $\mathsf{d}(w) = g$. Consider the choice function C^h induced by the following preferences:

$$\{w, x, z\} \succ \{w, \hat{z}\} \succ \{w, \hat{x}\} \succ \{w, x\} \succ \{w, z\} \succ \{w\} \succ$$
$$\{y, \hat{z}\} \succ \{y, x, z\} \succ \{y, \hat{x}\} \succ \{y, x\} \succ \{y, z\} \succ \{y\} \succ$$
$$\{x, z\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

Consider the offer process $(z, \hat{x}, \hat{z}, x, y, w)$—this offer process is observable, yet $\mathsf{d}(x) \notin C^{\hat{h}}(\{z, \hat{x}, \hat{z}, x, y\})$ while $\mathsf{d}(x) \in C^{\hat{h}}(\{z, \hat{x}, \hat{z}, x, y, w\})$. Hence, the choice function $C^{\hat{h}}$ is not observably substitutable across doctors.[25]

[25] It is also easy to see directly that it is not bilaterally substitutable: $x \notin C^{\hat{h}}(\{z, \hat{z}, x, y\})$ but $x \in C^{\hat{h}}(\{z, \hat{z}, x, y, w\})$.

However, when other hospitals have observably substitutable choice functions, a stable outcome always exists. To see this, let $\hat{C}^{\hat{h}}$ be the choice function induced by

$$\begin{aligned}\{w,x,z\} \succ \{w,\hat{z}\} \succ \{w,\hat{x}\} \succ \{w,x\} \succ \{w,z\} \succ \{w\} \succ \\ \{y,x,z\} \succ \{y,\hat{z}\} \succ \{y,\hat{x}\} \succ \{y,x\} \succ \{y,z\} \succ \{y\} \succ \\ \{x,z\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing;\end{aligned}$$

that is, consider the choice function $\hat{C}^h$ induced by switching the ordering of $\{y,x,z\}$ and $\{y,\hat{z}\}$ in the preferences above. This choice function for h is observably substitutable, and so running a cumulative offer process on $(\hat{C}^{\hat{h}}, (C^{\bar{h}})_{\bar{h} \in H \setminus \{\hat{h}\}})$ produces an outcome Y that is stable with respect to $(\hat{C}^{\hat{h}}, (C^{\bar{h}})_{\bar{h} \in H \setminus \{\hat{h}\}})$. We claim that Y is stable with respect to the original choice functions as well: Since Y is stable with respect to $(\hat{C}^{\hat{h}}, (C^{\bar{h}})_{\bar{h} \in H \setminus \{\hat{h}\}})$, the only way that Y could be unstable is if $\{y,x,z\} \subseteq Y$, and the only possible blocking set is $\{\hat{z}\}$. But if $\hat{z} \succ_{\mathsf{d}(z)} z$, then $\hat{z}$ is never rejected during any cumulative offer process. Hence $z \succ_{\mathsf{d}(z)} \hat{z}$, and so $\{\hat{z}\}$ is not a blocking set.

Example 5 shows that it is not sufficient to restrict attention to the cumulative offer mechanism in case one is only interested in obtaining a stable outcome.

5 Conclusion

In many real world settings, firms' preferences are not substitutable and yet stable and strategy-proof matching mechanisms exist—as demonstrated by Kamada and Kojima (2012, 2015), Sönmez (2013), Sönmez and Switzer (2013), and Dimakopoulos and Heller (2014). In fact, in all of the known applications of centralized matching under non-substitutable preferences, the cumulative offer mechanism is stable and strategy-proof. Our work shows that the ubiquity of cumulative offer mechanisms is not by chance: We show that when each hospital's choice function is observably substitutable, observably size monotonic, and non-manipulatable, the cumulative offer mechanism is the *unique* stable and strategy-proof mechanism. By contrast, if any of our three conditions fails, there exist unit-demand choice functions for the other hospitals such that *no* stable stable and strategy-proof mechanism exists. Thus, our results imply that the doctor-proposing cumulative offer process is an essential tool in the market designer's toolbox, as it is uniquely well-suited for many-to-one matching with contracts: whenever stable and strategy-proof matching is feasible, the cumulative offer mechanism is the unique mechanism that is stable and strategy-proof.

References

Aygün, O. and T. Sönmez (2012). Matching with contracts: The critical role of irrelevance of rejected contracts. Working Paper, Boston College.

Aygün, O. and T. Sönmez (2013). Matching with contracts: Comment. *American Economic Review 103*(5), 2050 – 2051.

Dimakopoulos, P. D. and C.-P. Heller (2014). Matching with waiting times: The german entry-level labour market for lawyers. Working Paper, Humboldt University of Berlin.

Flanagan, F. (2014). Relaxing the substitutes condition in matching markets with contracts. *Economics Letters 123*, 113 – 117.

Hatfield, J. W., N. Immorlica, and S. D. Kominers (2012). Testing substitutability. *Games and Economic Behavior 75*(2), 639–645.

Hatfield, J. W. and F. Kojima (2008). Matching with contracts: Comment. *American Economic Review 98*, 1189–1194.

Hatfield, J. W. and F. Kojima (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory 145*(5), 1704–1723.

Hatfield, J. W. and S. D. Kominers (2012). Matching in Networks with Bilateral Contracts. *American Economic Journal: Microeconomics 4*(1), 176–208.

Hatfield, J. W. and S. D. Kominers (2014). Contract design and stability in many-to-many matching. Working Paper, Harvard Business School.

Hatfield, J. W. and S. D. Kominers (2015). Hidden substitutes. Working paper, Harvard University.

Hatfield, J. W. and P. Milgrom (2005). Matching with contracts. *American Economic Review 95*, 913–935.

Hirata, D. and Y. Kasuya (2014). Cumulative offer process is order-independent. *Economics Letters 124*, 37–40.

Hirata, D. and Y. Kasuya (2015). On stable and strategy-proof rules in matching markets with contracts. Working paper, Harvard University.

Kadam, S. V. (2015). Unilateral substitutability implies substitutable completability in many-to-one matching with contracts. Working Paper, Harvard University.

Kamada, Y. and F. Kojima (2012). Stability and strategy-proofness for matching with constraints: A problem in the Japanese medical matching and its solution. *American Economic Review Papers & Proceedings 102*(3), 366–370.

Kamada, Y. and F. Kojima (2015). Efficient matching under distributional constraints: Theory and applications. *American Economic Review 105*, 67–99.

Kelso, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica 50*, 1483–1504.

Kominers, S. D. and T. Sönmez (Forthcoming, 2015). Matching with slot-specific priorities: Theory. *Theoretical Economics*.

Pycia, M. (2012). Stability and preference alignment in matching and coalition formation. *Econometrica 80*, 323–362.

Roth, A. E. (1982). The economics of matching: Stability and incentives. *Mathematics of Operations Research 7*, 617–628.

Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica 52*, 47–57.

Sönmez, T. (2013). Bidding for army career specialties: Improving the ROTC branching mechanism. *Journal of Political Economy 121*, 186–219.

Sönmez, T. and T. B. Switzer (2013). Matching with (branch-of-choice) contracts at United States Military Academy. *Econometrica 81*(2), 451–488.

A Proofs

We first gather some additional definitions that will be used throughout our proofs. We start by introducing more general notions of offer processes and observability. An *offer process* $\mathbf{x} = (x^1, \ldots, x^M)$ is a finite sequence of distinct contracts. Note that an offer process may contain contracts with many different hospitals. We denote by $\mathsf{c}(\mathbf{x}) \equiv \{x^1, \ldots, x^M\}$ the set of contracts offered during the offer process $\mathbf{x}$. Abusing notation slightly, we write $\mathbf{x} \in Y$ for some $Y \subseteq X$ if $\mathsf{c}(\mathbf{x}) \subseteq Y$. Fixing the choice functions of the hospitals, we say that an offer process is *observable* if $\mathsf{d}(x^m) \notin \mathsf{d}(C^H(\{x^1, \ldots, x^{m-1}\}))$ for all $m = 1, \ldots, M$. We use the term observable as, during a cumulative offer mechanism, only doctors who do not have contracts currently held by a hospital are allowed to make offers. Hence, an observable offer process is an offer process that could be generated by a cumulative offer mechanism. An offer process $\mathbf{x} = (x^1, \ldots, x^M)$ is *compatible* with a preference profile $\succ$ if

1. $\mathbf{x}$ is observable, and,
2. for all $m \in \{1, \ldots, M\}$,
    - $x^m \succ_{\mathsf{d}(x^m)} \emptyset$ and,
    - if $x \succ_{\mathsf{d}(x^m)} x^m$, then $x \in \{x^1, \ldots, x^{m-1}\}$.

An offer process $\mathbf{x}$ is *complete* with respect to $\succ$ and $C = (C^h)_{h \in H}$ if $\mathbf{x}$ is compatible with $\succ$ and, for all $d \notin \mathsf{d}(C^H(\mathsf{c}(\mathbf{x})))$, if $y \in X_d \smallsetminus \mathsf{c}(\mathbf{x})$, then $\emptyset \succ_d y$. We use the term complete as a cumulative offer mechanism ends only when every doctor is either employed or has proposed every acceptable contract. Finally, an offer process $\mathbf{y} = (y^1, \ldots, y^M)$ is *weakly observable* if, for all $m \leq M$, $\mathsf{d}(y^m) \notin \mathsf{d}(C^{\mathsf{h}(y^m)}(\{y^1, \ldots, y^{m-1}\}))$. Note that if $\mathbf{y}$ is weakly observable, then, for any $h \in H$, the offer process that is obtained from $\mathbf{y}$ by deleting all contracts that do not involve h is observable. In particular, if $\mathbf{y} = (y^1, \ldots, y^M)$ is weakly observable and C^h is observably substitutable for all $h \in H$, then $R^H(\{y^1, \ldots, y^{M-1}\}) \subseteq R^H(\mathsf{c}(\mathbf{y}))$.

Next, given a preference profile $\succ$ over $X \cup \{\emptyset\}$ and a set of contracts $Y \subseteq X$, we define the *restriction* $\succ^Y$ *of* $\succ$ *to* Y as follows:

1. For all $x, y \in Y$ such that $\mathsf{d}(x) = \mathsf{d}(y)$, $x \succ^Y_{\mathsf{d}(x)} y$ if and only if $x \succ_{\mathsf{d}(x)} y$.

2. For all $x \in X$, $x \succ^Y_{\mathsf{d}(x)} \emptyset$ if and only if $x \succ_{\mathsf{d}(x)} \emptyset$ and $x \in Y$.

Say that the preference profile $\succ$ *is consistent with* Y if the following conditions hold:

1. If $x \in Y$, then $x \succ_{\mathsf{d}(x)} \emptyset$, and

2. If $x \in X_{\mathsf{d}(x)} \smallsetminus Y$, then $\emptyset \succ_{\mathsf{d}(x)} x$.

In other words, the preference profile $\succ$ is consistent with Y if a contract y is acceptable to $\mathsf{d}(y)$ if and only if $y \in Y$. We also say that the preferences $\succ$ *are consistent with* $(\mathbf{y}, Y)$ if $\succ$ is consistent with Y and $\mathbf{y}$ is compatible with $\succ$.

An offer process $\mathbf{y} = (y^1, \ldots, y^M)$ is *weakly compatible with a preference profile* $\succ$, if, for all $m \in \{1, \ldots, M\}$, $h \in H$, and $d \in D$,

1. $y^m \in X_d$ implies that $y^m \succ_d \emptyset$ and

2. for any contract $y \in (X_h \cap X_d) \smallsetminus \{y^m\}$ such that $y \succ_d y^m$, $y \in \{y^1, \ldots, y^{m-1}\}$.

That is, an offer process $\mathbf{y}$ is weakly compatible with a preference profile $\succ$ if, for each $y^m \in \mathsf{c}(\mathbf{y})$,

1. y^m is an acceptable contract, and

2. the doctor making the offer y^m prefers y^m to every other contract *with the same hospital* that has not yet been offered.

We can write the combination of two offer processes $\mathbf{y} = (y^1, \ldots, y^M)$ and $\mathbf{z} = (z^1, \ldots, z^N)$ as $(\mathbf{y}, \mathbf{z}) = (w^1, \ldots, w^K)$ where

- $w^k = y^k$ for all $k \leq M$ and

- $w^k = z^{\ell_k}$ for $k > M$, where $\ell_k \equiv \min\{\ell \in 1, \ldots, N : z^\ell \notin \{w^1, \ldots, w^{k-1}\}\}$.

Our first lemma establishes a condition under which we can combine two different weakly observable offer processes to obtain another weakly observable offer process.

Lemma 1. *Suppose that the choice function of every hospital is observably substitutable across doctors. Let* $\mathbf{y}$ *and* $\mathbf{z}$ *be two weakly observable offer processes that are both weakly compatible with respect to the same preference profile* $\succ$*. Then* $(\mathbf{y}, \mathbf{z})$ *is a weakly observable offer process.*

Proof. Consider any weakly observable offer process $\mathbf{y} = (y^1, \ldots, y^M)$. We will prove the statement by induction on the length of $\mathbf{z} = (z^1, \ldots, z^N)$, showing at each step that $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{y})$ are weakly observable. If $N = 0$, the statement is trivially true. Hence, suppose that $(\mathbf{y}, (z^1, \ldots, z^{N-1}))$ and $((z^1, \ldots, z^{N-1}), \mathbf{y})$ are weakly observable.

We first show that $(\mathbf{y}, \mathbf{z})$ is weakly observable. There are two cases:

1. If $z^N \in \mathsf{c}(\mathbf{y})$, then $(\mathbf{y}, (z^1, \ldots, z^{N-1})) = (\mathbf{y}, \mathbf{z})$ and so $(\mathbf{y}, \mathbf{z})$ is weakly observable by the inductive assumption.

2. If $z^N \notin \mathsf{c}(\mathbf{y})$, we first note that $(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right) = \varnothing$;[26] that is, no contract between $\mathsf{d}(z^N)$ and $\mathsf{h}(z^N)$ is suggested in offer process $\mathbf{y}$ unless it was also suggested during $(z^1, \ldots, z^{N-1})$. Since $\mathbf{z}$ is weakly observable, we must have $\mathsf{d}(z^N) \notin \mathsf{d}(C^{\mathsf{h}(z^N)}(\{z^1, \ldots, z^{N-1}\}))$. By the inductive assumption, $((z^1, \ldots, z^{N-1}), \mathbf{y})$ is weakly observable. Since $C^{\mathsf{h}(z^N)}$ is observably substitutable across doctors, we then obtain that $\mathsf{d}(z^N) \notin \mathsf{d}(C^{\mathsf{h}(z^N)}(\{z^1, \ldots, z^{N-1}\} \cup \mathsf{c}(\mathbf{y})))$ given that $(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right) = \varnothing$; therefore, $(\mathbf{y}, \mathbf{z})$ is weakly observable by definition.

We now show by induction on m that, for all $m \leq M$, $(\mathbf{z}, (y^1, \ldots, y^m))$ is weakly observable. Suppose that for some $\bar{m} \leq M - 1$, the statement has already been shown for all $m' \leq \bar{m}$. We will show that the statement holds for $\bar{m} + 1$. There are two cases:

[26] Since $z^N \notin \mathsf{c}(\mathbf{y})$, we have that for all $z \in \mathsf{c}(\mathbf{y}) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right)$ it must be the case that $z \succ_{\mathsf{d}(z^N)} z^N$. Hence, if there existed $w \in (\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{z})) \cap \left(X_{\mathsf{d}(z^N)} \cap X_{\mathsf{h}(z^N)}\right)$, then $\mathbf{z}$ and $\mathbf{y}$ would not be weakly compatible with the same preference profile.

1. If $y^{\bar{m}+1} \in \mathsf{c}(\mathbf{z})$, then $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1})) = (\mathbf{z}, (y^1, \ldots, y^{\bar{m}}))$ and $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1}))$ is weakly observable by the inductive assumption.

2. If $y^{\bar{m}+1} \notin \mathsf{c}(\mathbf{z})$, we first note that $(\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right) = \varnothing$;[27] that is, no contract between $\mathsf{d}(y^{\bar{m}+1})$ and $\mathsf{h}(y^{\bar{m}+1})$ is suggested in offer process $\mathbf{z}$ unless it was also suggested during $\mathbf{y}$. Since $\mathbf{y}$ is weakly observable, we must have $\mathsf{d}(y^{\bar{m}+1}) \notin \mathsf{d}(C^{\mathsf{h}(y^{\bar{m}+1})}(\{y^1, \ldots, y^{\bar{m}}\}))$. We have already established that $((y^1, \ldots, y^{\bar{m}}), \mathbf{z})$ is weakly observable. Since $C^{\mathsf{h}(y^{\bar{m}+1})}$ is observably substitutable across doctors, we then obtain that $\mathsf{d}(y^{\bar{m}+1}) \notin \mathsf{d}(C^{\mathsf{h}(y^{\bar{m}+1})}(\{y^1, \ldots, y^{\bar{m}}\} \cup \mathsf{c}(\mathbf{z})))$ given that $(\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right) = \varnothing$; therefore, $(\mathbf{z}, (y^1, \ldots, y^{\bar{m}+1}))$ is weakly observable by definition.

This completes the proof of Lemma 1. $\square$

Our second preliminary Lemma derives a simple property of strategy-proof mechanisms.

Lemma 2. *Let $C = (C^h)_{h \in H}$ be a profile of choice functions and $\mathcal{M}$ be a strategy-proof mechanism for C. Let $Y \subseteq X$ be arbitrary and $\succ$ be a preference profile that is consistent with Y. Further suppose that $\mathcal{M}_d(\succ) = \{y\}$ for some doctor d and let $\hat{\succ} \equiv \succ^{\hat{Y}}$ for some set of contracts $\hat{Y} \subseteq Y$ such that $Y_{D \smallsetminus \{d\}} \cup \{y\} \subseteq \hat{Y}$. Then $\mathcal{M}_d(\hat{\succ}) = \{y\}$.*

Proof. First, note that $\hat{\succ}_{D \smallsetminus \{d\}} = \succ_{D \smallsetminus \{d\}}$. Suppose the conclusion of the theorem does not hold, and let $\hat{y} = \mathcal{M}_d(\hat{\succ}) \neq y$. If $\hat{y} \succ_d y$, then $\mathcal{M}$ is not strategy-proof, as $\mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}}) \succ_d \mathcal{M}(\succ)$. If $y \succ_d \hat{y}$, then $\mathcal{M}$ is not strategy-proof, as $\mathcal{M}(\succ) \hat{\succ}_d \mathcal{M}(\hat{\succ}_d, \succ_{D \smallsetminus \{d\}})$. $\square$

A.1 Proof of Theorem 1

For the proof of this Theorem, it is useful to introduce an alternative definition of observable substitutability that operates on sets of contracts.

[27] Since $y^{\bar{m}+1} \notin \mathsf{c}(\mathbf{z})$, we have that for all $z \in \mathsf{c}(\mathbf{y}) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right)$ it must be the case that $z \succ_{\mathsf{d}(y^{\bar{m}+1})} y^{\bar{m}+1}$. Hence, if there existed $w \in (\mathsf{c}(\mathbf{z}) \smallsetminus \mathsf{c}(\mathbf{y})) \cap \left(X_{\mathsf{d}(y^{\bar{m}+1})} \cap X_{\mathsf{h}(y^{\bar{m}+1})}\right)$, then $\mathbf{z}$ and $\mathbf{y}$ would not be weakly compatible with the same preference profile.

Definition 6. A set Y is *observably substitutable* under the choice profile $C = (C^h)_{h \in H}$ if, for any observable offer process $\mathbf{x} = (x^1, \ldots, x^M)$ such that $\mathsf{c}(\mathbf{x}) \subseteq Y$, we have that $R^H(\{x^1, \ldots, x^{M-1}\}) \subseteq R^H(\{x^1, \ldots, x^M\})$.

Note that a choice function C^h is *observably substitutable* according to Definition 2 if, and only if, X_h is observably substitutable under C^h according to Definition 6. Furthermore, note that if $Y \subseteq X$ is observably substitutable under $C = (C^h)_{h \in H}$, then any $Z \subseteq Y$ is also observably substitutable under $\{C^h\}_{h \in H}$.

It will also be helpful to define the *lower contour set of offer process* $\mathbf{y}$,

$$\mathsf{L}(\mathbf{y}) \equiv \{y^k \in \mathsf{c}(\mathbf{y}) : \nexists \hat{k} > k \text{ such that } \mathsf{d}(y^k) = \mathsf{d}(y^{\hat{k}})\},$$

that is, $\mathsf{L}(\mathbf{y})$ contains, for each doctor $d \in \mathsf{d}(\mathsf{c}(\mathbf{y}))$, the last contract in $\mathbf{y}$ that d is associated with.

The proof of Theorem 1 will rely on the following lemma, which we prove first.

Lemma 3. *Suppose that the mechanism $\mathcal{M}$ is stable and strategy-proof. Suppose that $Y \subseteq X$ is observably substitutable. Let $\succ$ be an arbitrary profile of preferences that is consistent with Y. If $\mathbf{y}$ is a complete offer process with respect to $\succ$, then $\mathcal{M}(\succ) = C^H(\mathsf{c}(\mathbf{y}))$ and $C^H(\mathsf{c}(\mathbf{y})) \subseteq \mathsf{L}(\mathbf{y})$.*

Proof. We proceed by induction on $M \equiv |Y|$. Our full inductive hypothesis is that for every preference profile $\succ$ consistent with Y, for any complete offer process $\mathbf{y}$ with respect to $\succ$,

1. $\mathcal{M}(\succ) = C^H(\mathsf{c}(\mathbf{y}))$, and
2. $\mathcal{M}(\succ) \subseteq \mathsf{L}(\mathbf{y})$.

The inductive hypothesis is clearly true for $M = 0$, that is, when $Y = \varnothing$. Now suppose it is true for all observably substitutable sets of size M or less. Now consider a set Y such that $|Y| = M + 1$. Consider any preference profile $\succ$ consistent with Y and any complete offer process $\mathbf{y} = (y^1, \ldots, y^N)$ with respect to $\succ$.

Observation 1. For each doctor d, we have that

$$\mathcal{M}_d(\succ) \in \{[\mathsf{L}(\mathbf{y})]_d, \varnothing\}.$$

Proof. Fix an arbitrary doctor $d \in D$. There are two cases:

1. $Y_d \smallsetminus \mathsf{c}(\mathbf{y}) \neq \varnothing$. Note first that $Y_d \smallsetminus \mathsf{c}(\mathbf{y}) \neq \varnothing$ implies $[C^H(\mathsf{c}(\mathbf{y}))]_d \neq \varnothing$. Furthermore, the assumption that Y is observably substitutable under $C = (C^h)_{h \in H}$ implies that $C^H(\mathsf{c}(\mathbf{y}))$ is a feasible outcome. Hence, there has to exist a unique contract $y \in [C^H(\mathsf{c}(\mathbf{y}))]_d$. Now let $\hat{Y} = Y \smallsetminus (Y_d \smallsetminus \mathsf{c}(\mathbf{y}))$ and $\hat{\succ} \equiv \succ^{\hat{Y}}$. Since $\mathbf{y}$ is a complete offer process with respect to $\hat{\succ}$ and $\hat{Y} \subsetneq Y$, the inductive hypothesis implies that $\mathcal{M}(\hat{\succ}) = C^H(\mathsf{c}(\mathbf{y}))$ and $C^H(\mathsf{c}(\mathbf{y})) \subseteq \mathsf{L}(\mathbf{y})$. In particular, $\{y\} = \mathcal{M}_d(\hat{\succ})$ and $y \in \mathsf{L}(\mathbf{y})$. If $\mathcal{M}_d(\succ) \in Y_d \smallsetminus \mathsf{c}(\mathbf{y})$, we obtain that $\mathcal{M}_d(\hat{\succ}) \succ_d \mathcal{M}(\succ)$ given that $\mathbf{y}$ is a complete offer process with respect to $\succ$. Hence, we must have $\mathcal{M}_d(\succ) \in \mathsf{c}(\mathbf{y})$. As $\mathcal{M}$ is strategy-proof, Lemma 2 implies that $\mathcal{M}_d(\hat{\succ}) = \mathcal{M}_d(\succ)$. Combining this last expression with the earlier observations that $\{y\} = \mathcal{M}_d(\hat{\succ})$ and $y \in \mathsf{L}(\mathbf{y})$ yields the desired result.

2. $Y_d \smallsetminus \mathsf{c}(\mathbf{y}) = \varnothing$. As $\mathcal{M}$ is individually rational, $\mathcal{M}_d(\succ) \subseteq \mathsf{c}(\mathbf{y})$. By way of contradiction, suppose that there exists a contract $\hat{y}$ such that $\{\hat{y}\} = \mathcal{M}_d(\succ)$ and $\hat{y} \succ_d [\mathsf{L}(\mathbf{y})]_d$.[28] Let $\hat{Y} = \{y \in Y : \mathsf{d}(y) \neq d \text{ or } y \succeq_d \hat{y}\}$; note that $|\hat{Y}| < |Y|$ as $\hat{y} \succ_d [\mathsf{L}(\mathbf{y})]_d$. Let $\hat{\succ} \equiv \succ^{\hat{Y}}$. As $\mathcal{M}$ is strategy-proof, Lemma 2 implies that $\mathcal{M}_d(\hat{\succ}) = \mathcal{M}_d(\succ) = \{\hat{y}\}$. Now, let $\bar{m} = \min\{m : \hat{y} \in R^H(\{y^1, \ldots, y^m\})\}$. It is clear that such an integer must exist since $\mathbf{y}$ is compatible with $\succ_d$ and $\mathsf{c}(\mathbf{y})$ contains the contract $[\mathsf{L}(\mathbf{y})]_d$ that d likes strictly less than $\hat{y}$. Construct a complete offer process $\mathbf{x} = (x^1, \ldots, x^{\bar{N}})$ with respect to $\hat{\succ}$ such that $x^n = y^n$ for all $n = 1, \ldots, \bar{m}$. Since $\hat{\succ}$ is consistent with $\hat{Y}$ and $|\hat{Y}| < |Y|$, the inductive assumption implies $\mathcal{M}(\hat{\succ}) \subseteq \mathsf{L}(\mathbf{x})$ and $\mathcal{M}(\hat{\succ}) = C^H(\mathsf{c}(\mathbf{x}))$. Since the set Y is observably substitutable under $\{C^h\}_{h \in H}$, we must have $\hat{y} \in R^H(\{x^1, \ldots, x^{\bar{m}}\})$. Therefore, we must have that $\hat{y} \notin C^H(\mathsf{c}(\mathbf{x})) = \mathcal{M}(\hat{\succ})$, contradicting our earlier conclusion that $\hat{y} = \mathcal{M}_d(\succ)$.

[28] Note that, by definition, $\mathsf{L}(\mathbf{y})$ contains at most one contract with each doctor.

This completes the proof of Observation 1. $\square$

Having proved the latter half of our inductive hypothesis on Y, i.e., that $\mathcal{M}(\succ) \subseteq \mathsf{L}(\mathbf{y})$, we now prove the former half, i.e., that $\mathcal{M}(\succ) = C^H(\mathsf{c}(\mathbf{y}))$. Suppose that $\mathcal{M}(\succ) \neq C^H(\mathsf{c}(\mathbf{y}))$. Then there exists a hospital h such that $\mathcal{M}_h(\succ) \neq C^h(\mathsf{c}(\mathbf{y}))$. Given that each $d \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})) \smallsetminus \mathcal{M}_h(\succ))$ strictly prefers $[C^h(\mathsf{c}(\mathbf{y}))]_d$ over $[\mathsf{L}(\mathbf{y})]_d$, $C^h(\mathsf{c}(\mathbf{y})) \smallsetminus \mathcal{M}_h(\succ)$ is a blocking set of $\mathcal{M}(\succ)$. Hence, $\mathcal{M}(\succ)$ cannot be stable, a contradiction. $\square$

With the help of Lemma 3 we will now prove Theorem 1. Suppose that the choice function of h is not observably substitutable. Let $\mathbf{y} = (y^1, \ldots, y^M)$ be an observable offer process such that $R^h(\{y^1, \ldots, y^{M-1}\}) \smallsetminus R^h(\{y^1, \ldots, y^M\}) \neq \varnothing$. Assume without loss of generality that $\mathbf{y}$ is a *minimal observable violation of substitutability* in the sense that every $Z \subsetneq \mathsf{c}(\mathbf{y})$ is observably substitutable under the choice profile $\{C^{\hat{h}}\}_{\hat{h} \in H}$.

Claim 1. $C^h(\mathsf{c}(\mathbf{y})) \subseteq \mathsf{L}(\mathbf{y})$.

Proof. We show first that, for all preference profiles $\succ$ consistent with $(\mathbf{y}, \mathsf{c}(\mathbf{y}))$, $\mathcal{M}(\succ) \subseteq \mathsf{L}(\mathbf{y})$. Suppose, by way of contradiction, that there exists a preference profile $\succ$ consistent with $(\mathbf{y}, \mathsf{c}(\mathbf{y}))$ such that $\mathcal{M}(\succ) \not\subseteq \mathsf{L}(\mathbf{y})$. Let $\hat{y}$ be an arbitrary element of $\mathcal{M}(\succ) \smallsetminus \mathsf{L}(\mathbf{y})$ and let $\hat{d} \equiv \mathsf{d}(\hat{y})$. Note that $\hat{y} \in \mathcal{M}(\succ) \smallsetminus \mathsf{L}(\mathbf{y})$ implies that there exists a contract $\tilde{y} \in [\mathsf{L}(\mathbf{y})]_{\hat{d}}$ such that $\hat{y} \succ_{\hat{d}} \tilde{y}$. Let $\hat{Y} = \mathsf{c}(\mathbf{y}) \smallsetminus \{\tilde{y}\}$ and $\hat{\succ} = \succ^{\hat{Y}}$. Since $\mathcal{M}$ is strategy-proof, Lemma 2 implies that $\hat{y} \in \mathcal{M}(\hat{\succ})$.

Now, let $\bar{m} = \min\{m : \hat{y} \in R^H(\{y^1, \ldots, y^m\})\}$; such an $\bar{m}$ must exist given that $\hat{d}$ proposes $\tilde{y}$ along $\mathbf{y}$ and $\hat{y} \succ_{\hat{d}} \tilde{y}$. Let $\mathbf{x} = (x^1, \ldots, x^N)$ be a complete offer process with respect to $\hat{\succ}$ such that $x^n = y^n$ for all $n = 1, \ldots, \bar{m}$. Note that $\hat{y} \notin C^H(\mathsf{c}(\mathbf{x}))$ since $\hat{y} \in R^H(\{x^1, \ldots, x^{\bar{m}}\})$, $\hat{Y}$ is observably substitutable[29], and $\mathbf{x}$ is observable. Moreover, by Lemma 3, $\mathcal{M}(\hat{\succ}) = C^H(\mathsf{c}(\mathbf{x}))$. Hence, $\hat{y} \notin \mathcal{M}(\hat{\succ})$, contradicting our earlier conclusion that $\hat{y} \in \mathcal{M}(\hat{\succ})$. This shows that we must have $\mathcal{M}(\succ) \subseteq \mathsf{L}(\mathbf{y})$.

[29]The observable subsitutability of $\hat{Y}$ follows from the fact that $\mathbf{y}$ is a minimal observation of substitutability.

Now, suppose by way of contradiction that $C^h(\mathsf{c}(\mathbf{y})) \nsubseteq \mathsf{L}(\mathbf{y})$. If $C^h(\mathsf{c}(\mathbf{y})) \nsubseteq \mathsf{L}(\mathbf{y})$, then $\mathcal{M}(\succ)$ is blocked by $C^h(\mathsf{c}(\mathbf{y})) \smallsetminus \mathcal{M}(\succ)$, contradicting the stability of $\mathcal{M}$. $\square$

We let $\hat{y} \in R^h(\{y^1, \ldots, y^{N-1}\}) \smallsetminus R^h(\{y^1, \ldots, y^N\})$ be arbitrary, and note that $\hat{y} \in R^h(\{y^1, \ldots, y^{N-1}\}) \smallsetminus R^h(\{y^1, \ldots, y^N\})$ implies that $C^h(\{y^1, \ldots, y^N\}) \neq C^h(\{y^1, \ldots, y^{N-1}\})$. By irrelevance of rejected contracts, the last statement requires that $y^N \in C^h(\{y^1, \ldots, y^N\})$. Since $\mathbf{y}$ is observable and $y^N \in C^h(\{y^1, \ldots, y^N\})$, we must have $\mathsf{d}(\hat{y}) \neq \mathsf{d}(y^N)$ as no hospital ever chooses two contracts with the same doctor. We claim that $\mathsf{d}(\hat{y}) \notin \mathsf{d}(C^H(\{y^1, \ldots, y^{N-1}\}))$. To see this, note first that Claim 1 and $\hat{y} \in C^H(\mathsf{c}(\mathbf{y}))$ imply that $\hat{y} \in \mathsf{L}(\mathbf{y})$. Furthermore, since $\mathbf{y}$ is a minimal observable violation of substitutability, it has to be the case that $C^h(\{y^1, \ldots, y^{N-1}\}) \subseteq \mathsf{L}((y^1, \ldots, y^{N-1}))$. Since $\mathsf{d}(\hat{y}) \neq \mathsf{d}(y^N)$, we have that $[\mathsf{L}((y^1, \ldots, y^{N-1}))]_{\mathsf{d}(\hat{y})} = [\mathsf{L}(\mathbf{y})]_{\mathsf{d}(\hat{y})}$ so that $C^h(\{y^1, \ldots, y^{N-1}\}) \cap X_{\mathsf{d}(\hat{y})} \subseteq \{\hat{y}\}$. Since $\hat{y} \in R^h(\{y^1, \ldots, y^{N-1}\})$, we obtain the desired statement.

Now, h' be another hospital, let $\bar{y}'$ be a contract between h' and $\mathsf{d}(y^N) \equiv \bar{d}$, and let $\hat{y}'$ be a contract between h' and $\hat{d}$. Let the choice function of h' be given by

$$C^{h'}(Z) = \begin{cases} \{\hat{y}'\} & \hat{y}' \in Z \\ \{\bar{y}'\} & \hat{y}' \notin Z \text{ and } \bar{y} \in Z \\ \varnothing & \text{otherwise.} \end{cases}$$

Let $\succ$ be a preference profile that is consistent with $(\mathbf{y}, \mathsf{c}(\mathbf{y}) \cup \{\bar{y}'\})$ such that $y \succ_{\bar{d}} \bar{y}'$, for all $y \in \mathsf{c}(\mathbf{y}) \smallsetminus \{y^N\}$, and $\bar{y} \mathrel{\hat{\succ}_{\bar{d}}} y^N$. A straightforward variation of the arguments used in the proof of Claim 1 shows that we must have $\mathcal{M}(\succ) \subseteq \mathsf{L}(\mathbf{y}) \cup \{\bar{y}'\}$.[30] By stability, this implies that $\bar{y}' \in \mathcal{M}(\succ)$ and therefore $y^N \notin \mathcal{M}(\succ)$. Another application of stability yields $\mathcal{M}_h(\succ) = C^h(\{y^1, \ldots, y^{N-1}\}) \subseteq \mathsf{L}((y^1, \ldots, y^{N-1}))$. Since $\mathbf{y}$ is a minimal observable violation of substitutability, $\hat{y} \notin \mathcal{M}_h(\succ)$ and $\mathcal{M}_{\hat{d}}(\succ) = \varnothing$.

[30]Suppose to the contrary that there exists a contract $\hat{y} \in \mathcal{M}(\succ) \smallsetminus (\mathsf{L}(\mathbf{y}) \cup \{\bar{y}'\})$. Letting $\hat{Y} \equiv \{y \in Y : \mathsf{d}(y) \neq \mathsf{d}(\hat{y}) \text{ or } y \succeq_{\mathsf{d}(\hat{y})} \hat{y}\}$ and $\mathbf{x}$ be a complete offer process with respect to $\hat{\succ} \equiv \succ^{\hat{Y}}$. Lemma 3 implies that $\mathcal{M}_h(\hat{\succ}) \subseteq \mathsf{L}(\mathbf{x})$. Since $\hat{y} \in \mathcal{M}(\succ) \smallsetminus (\mathsf{L}(\mathbf{y}) \cup \{\bar{y}\})$, observability of $\mathbf{y}$ implies that $\hat{y} \notin \mathcal{M}(\hat{\succ})$, a contradiction. We omit the remaining details.

Now consider a preference profile $\hat{\succ}$ such that

1. $\hat{\succ}_{-\hat{d}} = \succ_{-\hat{d}}$,
2. for all $y, z \in [\mathsf{c}(\mathbf{y})]_{\hat{d}}$, $y \ \hat{\succ}_{\hat{d}} \ z$ if and only if $y \ \succ_{\hat{d}} \ z$, and
3. $\hat{y}' \ \hat{\succ}_{\hat{d}} \ \emptyset$ and, for all $y \in [\mathsf{c}(\mathbf{y})]_{\hat{d}}$, $y \ \hat{\succ}_{\hat{d}} \ \hat{y}'$.

By strategy-proofness, we must have $\mathcal{M}_{\hat{d}}(\hat{\succ}) \in \{\varnothing, \{\hat{y}'\}\}$. Stability then implies that $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \{\hat{y}'\}$. Again, a straightforward variation of the arguments used in the proof of Claim 1 shows that we must have $\mathcal{M}_h(\hat{\succ}) \subseteq \mathsf{L}(\mathbf{y})$. In particular, all doctors weakly prefer their contract in $\mathsf{L}(\mathbf{y})$ over the contract in $\mathcal{M}_h(\hat{\succ})$. Since at least $\hat{d}$ strictly prefers $[\mathsf{L}(\mathbf{y})]_{\hat{d}} = \{\hat{y}\}$ over $\mathcal{M}_{\hat{d}}(\hat{\succ}) = \{\hat{y}'\}$, $\mathcal{M}(\hat{\succ})$ is blocked by $C^h(\mathsf{c}(\mathbf{y})) \smallsetminus \mathcal{M}(\hat{\succ})$, contradicting the stability of $\mathcal{M}$.

A.2 Proof of Proposition 2

Let $\mathcal{M}$ be an arbitrary stable and strategy-proof mechanism. Fix a preference profile $\succ$ and a complete offer process $\mathbf{x}$. By Lemma 3, we must have $\mathcal{M}(\succ) = C^H(\mathsf{c}(\mathbf{x}))$. By Proposition 1, $C^H(\mathsf{c}(\mathbf{x})) = C^H(\mathsf{c}(\mathbf{y}))$ for any complete offer process $\mathbf{y}$ with respect to $\succ$. This completes the proof.

A.3 Proof of Proposition 4

Fix a profile of choice functions $C = (C^h)_{h \in H}$ that are observably substitutable. Let $\succ$ be an arbitrary preference profile for the doctors and $d \in D$ be an arbitrary doctor. Let $\mathbf{x} = (x^1, \dots, x^M)$ be a complete offer process with respect to $\succ$. By Lemma 3, we must have that $\mathcal{C}(\succ) = C^H(\mathsf{c}(\mathbf{x}))$ and $C^H(\mathsf{c}(\mathbf{x})) \subseteq \mathsf{L}(\mathbf{x})$. Let $\hat{\succ}_d$ be an arbitrary truncation of $\succ_d$ and y be the least preferred acceptable contract for d according to $\hat{\succ}_d$. If $[C^H(\mathsf{c}(\mathbf{x}))]_d \succ_d y$, it is easy to see that $\mathbf{x}$ is a complete offer process with respect to $(\hat{\succ}_d, \succ_{-d})$ and hence $\mathcal{C}(\hat{\succ}) = \mathcal{C}(\succ)$. So assume that $y \succ_d [C^H(\mathsf{c}(\mathbf{x}))]_d$. Since $\mathbf{x}$ is observable and $y \succ_d [C^H(\mathsf{c}(\mathbf{x}))]_d$, there must exist a smallest integer $\bar{m} \leq M$ such that $y \in R^H(\{x^1, \ldots, x^{\bar{m}}\})$. Since $\hat{\succ}_d$ is a truncation of $\succ_d$, there exists a complete offer process $\mathbf{y} = (y^1, \ldots, y^N)$ at $\hat{\succ}$ such that, for all $n \leq \bar{m}$, $y^n = x^n$. Since all choice functions are observably substitutable and since $y \in R^H(\{x^1, \ldots, x^{\bar{m}}\})$, we must have $y \in R^H(\mathsf{c}(\mathbf{y}))$. Since y is the least preferred contract with respect to $\hat{\succ}_d$ and since $\mathbf{y}$ is observable, we must have $[C^H(\mathsf{c}(\mathbf{y}))]_d = \varnothing$. By Lemma 3, we must have that $\mathcal{C}(\hat{\succ}_d, \succ_{-d}) = C^H(\mathsf{c}(\mathbf{y}))$ and hence also $[\mathcal{C}(\hat{\succ}_d, \succ_{-d})]_d = \varnothing$. Since $\mathcal{C}(\succ)$ is individually rational for all doctors, we obtain that $\mathcal{C}(\succ) \succeq_d \mathcal{C}(\hat{\succ}_d, \succ_{-d})$. Since $\succ$, $d \in D$, and $\hat{\succ}_d$ were all arbitrary, this completes the proof of Proposition 4.

A.4 Proof of Theorem 2

The proof of Theorem 2 will rely on the following lemma, which we prove first.

Lemma 4. *Suppose that C^h is observably substitutable but not observably size monotonic. Then there exists a contract x and a set Y such that $\mathsf{d}(x) \notin \mathsf{d}(Y)$, for all $d \in D$, $|Y_d| \leq 1$, and $|C^h(Y \cup \{x\})| < |C^h(Y)|$.*

Proof. Since the choice function of h is not observably size monotonic, we have an observable offer process $(x^1, \ldots, x^M)$ such that $|C^h(\{x^1, \ldots, x^{M-1}\})| > |C^h(\{x^1, \ldots, x^M\})|$. Since C^h is observably substitutable, $C^h(\{x^1, \ldots, x^M\}) \subseteq \{x^M\} \cup C^h(\{x^1, \ldots, x^{M-1}\})$. Let $Y = C^h(\{x^1, \ldots, x^{M-1}\})$; note that $C^h(Y) = Y$ and $C^h(\{x^1, \ldots, x^M\}) = C^h(Y \cup \{x^M\})$ by the irrelevance of rejected contracts condition. Moreover, since each hospital chooses at most one contract with each doctor, $|Y_d| = |[C^h(Y)]_d| = 1$ for all $d \in D$.

Finally, since $(x^1, \ldots, x^M)$ is observable, $\mathsf{d}(x^M) \notin C^h(\{x^1, \ldots, x^{M-1}\}) = Y$. Hence, setting $x = x^M$ completes the construction. □

With the help of Lemma 4, the proof now proceeds analogously to the proof of Theorem 9 in Hatfield and Milgrom (2005). Since the choice function of h is not observably size monotonic, by Lemma 4 we have a contract x and a set Y such that $\mathsf{d}(x) \notin Y$, for all $d \in D$,

$|Y_d| \leq 1$, and $|C^h(Y \cup \{x\})| < |C^h(Y)|$. Let $\{y, z\} \subseteq C^h(Y) \smallsetminus C^h(Y \cup \{x\})$; note that $\mathsf{d}(x)$, $\mathsf{d}(y)$, and $\mathsf{d}(z)$ are all distinct doctors.

Let $\bar{x}$ be a contract between $\mathsf{d}(x)$ and $\bar{h} \neq h$, $\bar{y}$ be a contract between $\mathsf{d}(y)$ and $\bar{h}$, and $\bar{z}$ be a contract between $\mathsf{d}(z)$ and $\bar{h}$.

We now define the preferences of the doctors as:

1. For every doctor $d \in \mathsf{d}(Y) \smallsetminus \{\mathsf{d}(y), \mathsf{d}(z)\}$, setting y^d to be the unique contract in Y such that $\mathsf{d}(y^d) = d$, let

$$\succ_d : y^d \succ \emptyset;$$

2. For $\mathsf{d}(x)$, let

$$\succ_{\mathsf{d}(x)} : \bar{x} \succ x \succ \emptyset;$$

3. For $\mathsf{d}(y)$, let

$$\succ_{\mathsf{d}(y)} : y \succ \bar{y} \succ \emptyset;$$

4. For $\mathsf{d}(z)$, let

$$\succ_{\mathsf{d}(z)} : \bar{z} \succ z \succ \emptyset.$$

Finally, we define the choice function of $\bar{h}$ as

$$C^{\bar{h}}(Z) = \begin{cases} \{\bar{y}\} & \bar{y} \in Z \\ \{\bar{z}\} & \bar{y} \notin Z \text{ and } \bar{z} \in Z \\ \{\bar{x}\} & \bar{y}, \bar{z} \notin Z \text{ and } \bar{x} \in Z \\ \varnothing & \text{otherwise.} \end{cases}$$

The only stable outcome under these choice functions is $C^h(Y \cup \{x\}) \cup \{\bar{y}\}$, under which $\mathsf{d}(z)$ is unemployed. However, if $\mathsf{d}(z)$ reports his preferences as

$$\hat{\succ}_{\mathsf{d}(z)} : z \succ \emptyset$$

then the only stable outcome is $Y \cup \{\bar{x}\}$, under which $\mathsf{d}(z)$ obtains z and, hence, is strictly better off.

A.5 Proof of Theorem 4

By Proposition 3, which does not rely on Theorem 4, observable substitutability is sufficient for the cumulative offer mechanism to produce a stable outcome. Hence, we only need to establish that observable substitutability, observable size monotonicity, and non-manipulatability imply that the cumulative offer mechanism is strategy-proof.[31]

Consider a profile of choice functions $C = (C^h)_{h \in H}$ such that, for each $h \in H$, C^h is observably substitutable and observably size monotonic. Suppose that the cumulative offer mechanism is not strategy-proof, so that there exists a preference profile $\succ$, a doctor $\hat{d}$, and a preference relation $\hat{\succ}_{\hat{d}}$ for that doctor such that $\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}}) \succ_{\hat{d}} \mathcal{C}(\succ)$. Let $\hat{x} \in [\mathcal{C}(\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}})]_{\hat{d}}$ be the contract that $\hat{d}$ obtains under $\hat{\succ} \equiv (\hat{\succ}_{\hat{d}}, \succ_{D \setminus \{\hat{d}\}})$ and let $\hat{h} \equiv \mathsf{h}(\hat{x})$. We will show that $C^{\hat{h}}$ is manipulatable.

As a first step of the proof, we will modify the preference profiles $\succ$ and $\hat{\succ}$. Let $\mathbf{x} = (x^1, \ldots, x^K)$ be a complete offer process with respect to $\succ$ and $\hat{\mathbf{x}}$ be a complete offer process with respect to $\hat{\succ}$. Note that $\mathcal{C}(\succ) = C^H(\mathsf{c}(\mathbf{x}))$ and $\mathcal{C}(\hat{\succ}) = C^H(\mathsf{c}(\hat{\mathbf{x}}))$ by Proposition 1. It is without loss of generality to assume that all contracts in $X \setminus (\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}}))$ are unacceptable to the associated doctors under $\succ$ and $\hat{\succ}$.[32] Furthermore, it is also without loss of generality to assume that $\hat{x}$ is the lowest ranked acceptable contract under $\succ_{\hat{d}}$ and $\hat{\succ}_{\hat{d}}$.[33] Finally, note that by Proposition 1 we can assume without loss of generality that $\mathbf{x}$ is

[31] As we show in Appendix B.1, irrelevance of rejected contracts is necessary for the stability of the cumulative offer mechanism. Our proof that the cumulative offer mechanism is strategy-proof when choice functions are observably substitutable, observably size monotonic, and non-manipulatable does not depend on the irrelevance of rejected contracts condition.

[32] Clearly, $\mathbf{x}$ is a complete offer process with respect to $\succ^{\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})}$; hence, by Proposition 1, $\mathcal{C}(\succ^{\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})}) = C^H(\mathsf{c}(\mathbf{x})) = \mathcal{C}(\succ)$. Similarly, $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}^{\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})}$, and so by Proposition 1, $\mathcal{C}(\hat{\succ}^{\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})}) = C^H(\mathsf{c}(\hat{\mathbf{x}})) = \mathcal{C}(\hat{\succ})$.

[33] For $\hat{\succ}_{\hat{d}}$, the statement follows immediately since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}^{X \setminus \{y \in X_{\hat{d}} : \hat{x} \hat{\succ}_{\hat{d}} y\}}$. To see that the statement is also true for $\succ_{\hat{d}}$, let $x^1, \ldots, x^M \in X$ be contracts such that $\mathbf{x} = (x^1, \ldots, x^M)$. The assumption that $\hat{x} \succ_{\hat{d}} \mathcal{C}(\succ)$ implies that there exists an integer $\bar{m} = \min\{m : \hat{x} \in R^H(\{x^1, \ldots, x^m\})\}$. Now consider an ordering $\vdash$ such that $x^m \vdash x^{m+1}$, for all

the offer process with respect to an ordering $\vdash$ such that, for all $x \in X \setminus X_{\hat{d}}$ and all $y \in X_{\hat{d}}$, $x \vdash y$. This implies that the cumulative offer process corresponding to $\mathbf{x}$ ends with the rejection of $\hat{x}$, i.e., that $\hat{x}$ is the unique element of $R^H(\{x^1, \ldots, x^K\}) \setminus R^H(\{x^1, \ldots, x^{K-1}\})$.[34]

Now set $\succ' \equiv \succ^{X_{\hat{h}}}$ and $\hat{\succ}' \equiv \hat{\succ}^{X_{\hat{h}}}$. Let $\mathbf{x}'$ be a complete offer process with respect to $\succ'$, and let $\hat{\mathbf{x}}'$ be a complete offer process with respect to $\hat{\succ}'$. By Proposition 1, we must have that $\mathcal{C}\succ' = C^H(\mathsf{c}(\mathbf{x}'))$ and $\mathcal{C}\hat{\succ}' = C^H(\mathsf{c}(\hat{\mathbf{x}}'))$. To show that the preferences of $\hat{h}$ are manipulatable, it is thus sufficient to establish that $\hat{x} \in C^{\hat{h}}(\mathsf{c}(\hat{\mathbf{x}}'))$ and $\hat{x} \in R^{\hat{h}}(\mathsf{c}(\mathbf{x}'))$. To see that the latter statement is true, let $x^1, \ldots, x^M \in X_{\hat{h}}$ be contracts such that $(x^1, \ldots, x^M)$ is the subsequence of $\mathbf{x}$ that consists of all and only contracts with $\hat{h}$. Let $\bar{m} = \min\{m : \hat{x} \in R^{\hat{h}}(\{x^1, \ldots, x^m\})\}$. Since $\hat{x} \succ_{\hat{d}} \mathcal{C}(\succ)$, the definition of a cumulative offer process implies that such an integer has to exist. Now consider an ordering $\vdash$ such that $x^m \vdash x^{m+1}$, for all $m \in \{1, \ldots, M-1\}$, and $x^M \vdash y$, for all $y \in X \setminus \{x^1, \ldots, x^M\}$. By the construction of $\succ'$, the first $\bar{m}$ contracts in the complete offer process with respect to $\succ'$ and $\vdash$ are $x^1, \ldots, x^{\bar{m}}$. Given that $C^{\hat{h}}$ is observably substitutable and $\hat{x} \in R^{\hat{h}}(\{x^1, \ldots, x^{\bar{m}}\})$, $\hat{x}$ must be rejected by $\hat{h}$ when $\hat{h}$ has access to all contracts in the complete offer process with respect to $\succ'$ and $\vdash$. By Proposition 1, this implies $\hat{x} \in R^{\hat{h}}(\mathsf{c}(\mathbf{x}'))$. Since $\hat{x}$ is the least-preferred acceptable contract for doctor $\hat{d}$ under $\succ'$, this implies that $\varnothing = \mathcal{C}_{\hat{d}}(\succ')$.

Claim 2. $\hat{x} \in C^{\hat{h}}(\mathsf{c}(\hat{\mathbf{x}}'))$.

Claim 2 suffices to show the result as $\mathbf{x}'$ is a complete offer process with respect to $\succ'$ and $\hat{\mathbf{x}}'$ is a complete offer process with respect to $\hat{\succ}'$, and thus $\hat{x} = \mathcal{C}_{\hat{d}}(\hat{\succ}') \succ'_{\hat{d}} \mathcal{C}_{\hat{d}}(\succ') = \emptyset$ implies that $C^{\hat{h}}$ is manipulatable.

Before proving Claim 2, we depart from the specific setting of our proof to introduce some important auxiliary concepts. Consider an arbitrary preference profile $\tilde{\succ}$ and an arbitrary

$m \in \{1, \ldots, M-1\}$, and $x^M \vdash y$, for all $y \in X \setminus \{x^1, \ldots, x^M\}$. It is clear that $(x^1, \ldots, x^{\bar{m}})$ is a part of a complete offer process with respect to $\vdash$ and $\succ^{X \setminus \{y \in X_{\hat{d}} : \hat{x} \succ_{\hat{d}} y\}}$. Observable substitutability implies that $\hat{x} \notin \mathcal{C}(\succ^{X \setminus \{y \in X_{\hat{d}} : \hat{x} \succ_{\hat{d}} y\}})$.

[34] To see this, note that by observable size monotonicity at most one contract is rejected in each step of the cumulative offer process with respect to $\succ$ and $\vdash$. Since $\hat{x}$ is the least preferred contract with respect to $\succ_{\hat{d}}$, the cumulative offer process with respect to $\succ$ and $\vdash$ ends as soon as $\hat{x}$ is rejected.

offer process $\mathbf{z}$. A *pre-run rejection chain* at $\mathbf{z}$ is a (non-empty) sequence of contracts $\mathbf{y} = (y^1, \ldots, y^N)$ such that the following conditions are satisfied:

1. For doctor $d^1 \equiv \mathsf{d}(y^1)$,

   (a) $d^1 \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{z})))$,

   (b) $d^1 \notin \mathsf{d}(C^{\mathsf{h}(y^1)}(\mathsf{c}(\mathbf{z})))$, and

   (c) for all $y \in [(X_{\mathsf{h}(y^1)} \cap X_{d^1}) \cup \{\emptyset\}] \smallsetminus \mathsf{c}(\mathbf{z})$, $y^1 \tilde{\succ}_{d^1} y$.

2. For all $n \in \{2, \ldots, N\}$, for doctor $d^n \equiv \mathsf{d}(y^n)$,

   (a) $d^n \neq d^1$,

   (b) $d^n \notin \mathsf{d}(C^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}))$,

   (c) $d^n \in \mathsf{d}(R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}) \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-2}\}))$, and

   (d) for all $y \in (X_{d^n} \smallsetminus (\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}) \cup \{\emptyset\}$, $y^n \tilde{\succ}_{d^n} y$.

3. $d^1 \in \mathsf{d}(R^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y})) \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{N-1})\}))$.

Essentially, a pre-run rejection chain is a chain started by a doctor who is currently employed at some hospital making an offer to a hospital different from the one that currently employs him (Condition 1). That other hospital then rejects a currently-held contract, inducing the doctor associated with that contact to make a new offer, and so on (Condition 2). This process continues until the originally-proposing doctor has a contract rejected (Condition 3). Note that, for all $n \geq 2$, $X_{\mathsf{d}(y^n)} \smallsetminus (\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}) = X_{\mathsf{d}(y^n)} \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$ since $\mathsf{d}(y^n) \notin \mathsf{d}(C^H(\mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\}))$. Hence, the only point in a pre-run rejection at which a doctor might propose a contract that is not that doctor's favorite contract among all contracts that have not been rejected yet is at the beginning of the pre-run rejection chain. Note that if $\mathbf{z}$ is weakly observable and weakly compatible with $\tilde{\succ}$, then $(\mathbf{z}, \mathbf{y})$ is weakly observable and weakly compatible with respect to $\tilde{\succ}$ when $\mathbf{y}$ is a pre-run rejection chain at $\mathbf{z}$. Furthermore, if $\mathbf{z}$ is such that $C^H(\mathsf{c}(\mathbf{z}))$ is a feasible outcome and $\mathbf{y}$ is a pre-run rejection chain at $\mathbf{z}$, then $C^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ is a feasible outcome. In particular, there exists a unique contract $\tilde{y} \in [C^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))]_{d^1}$. From the definition of a pre-run rejection chain it follows that $\tilde{y}$ must be the highest ranking acceptable contract in $(X_{d^1} \cap X_{\mathsf{h}(y^1)}) \smallsetminus R^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ with respect to $\tilde{\succ}_{d^1}$. However, note that there might still be contracts $\hat{y} \in X_{d^1} \smallsetminus (R^H(\mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y})) \cup X_{\mathsf{h}(y^1)})$ such that $\hat{y} \mathrel{\tilde{\succ}_{d^1}} \tilde{y}$.

A *generalized pre-run rejection chain* at $\mathbf{z}$ is an offer process $\mathbf{y} = (\mathbf{y}^1, \ldots, \mathbf{y}^L)$ such that for each $\ell \in \{1, \ldots, L\}$, $\mathbf{y}^\ell$ is a pre-run rejection chain at $(\mathbf{z}, \mathbf{y}^1, \ldots, \mathbf{y}^{\ell-1})$. An offer process $\mathbf{w}$ can be *obtained from* $\mathbf{z}$ *by pre-running rejection chains* if $\mathbf{w} = (\mathbf{z}, \mathbf{y})$ for some generalized pre-run rejection chain $\mathbf{y}$ at $\mathbf{z}$.

Proof of Claim 2. Let $\check{\mathbf{x}}$ be a complete offer process with respect to $\succ_{-\hat{d}}$. Note that $\mathsf{c}(\check{\mathbf{x}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$. This follows from Proposition 1 since any complete offer process for $\succ$ and $\hat{\succ}$ has to contain all contracts that are contained in a complete offer process with respect to an ordering $\vdash$ such that, for all $y \in X \smallsetminus X_{\hat{d}}$ and all $x \in X_{\hat{d}}$, $y \vdash x$. The key step of our proof lies in the construction of an offer process that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains and that satisfies four specific properties.

Claim 3. *There exists an offer process* $\mathbf{y}^*$ *such that*

1. $\mathbf{y}^*$ *can be obtained from* $\check{\mathbf{x}}$ *by pre-running rejection chains,*

2. $\mathsf{c}(\mathbf{y}^*) \subseteq X \smallsetminus X_{\hat{d}}$,

3. $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{y}^*)$, *and*

4. $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{y}^*))$.

Condition 1 ensures in particular that $\mathbf{y}^*$ is weakly observable; Condition 2 requires that no contract in $\mathsf{c}(\mathbf{y}^*)$ names doctor $\hat{d}$; Condition 3 ensures that $\mathsf{c}(\mathbf{y}^*)$ contains all the contracts that are proposed in the cumulative offer process for $\hat{\succ}'$ that are *not* in the cumulative offer process for $\hat{\succ}$; Condition 4 ensures that all rejections that occur when contracts in $\mathsf{c}(\mathbf{y}^*)$

become available to hospitals in addition to contracts in $\mathsf{c}(\hat{\mathbf{x}})$ concern contracts that are already rejected when hospitals have access to contracts in $\mathsf{c}(\mathbf{y}^*) \subseteq X \setminus X_{\hat{d}}$.

Before proceeding to the proof of Claim 3, we argue why it implies Claim 2 that $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}'))$. Take an offer process $\mathbf{y}^*$ that satisfies the four conditions of Claim 3. By the fourth condition, $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{y}^*))$. Since $\mathsf{c}(\mathbf{y}^*) \subseteq X \setminus X_{\hat{d}}$ by the second condition, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq X \setminus X_{\hat{d}}$. Given that $\hat{x} \in [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\hat{d}}$, we obtain $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$. Since $\mathbf{y}^*$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains by the first condition, $\mathbf{y}^*$ is weakly observable and weakly compatible with $\hat{\succ}$. Since $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ are also both weakly observable and weakly compatible with $\hat{\succ}$, $(\hat{\mathbf{x}}', \hat{\mathbf{x}}, \mathbf{y}^*)$ is weakly observable by Lemma 1. Since there are no observable violations of substitutes, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$. By the third condition of Claim 3, $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{y}^*)$ and thus $\mathsf{c}(\hat{\mathbf{x}}') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)$. In particular, $R^H(\mathsf{c}(\hat{\mathbf{x}}') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*)) = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{y}^*))$ and we obtain an observable violation of substitutability if $\hat{x} \in R^H(\mathsf{c}(\hat{\mathbf{x}}'))$. Hence, $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}'))$.

Proof of Claim 3. In the proof of Claim 3, we will iteratively construct an offer process $\mathbf{y}^*$ that satisfies Conditions 1–4 of Claim 3 starting at $\check{\mathbf{x}}$. A key step of the construction involves extending a given generalized pre-run rejection chain at $\check{\mathbf{x}}$. The next claim provides a simple condition under which such an extension is possible.

Claim 4. *Let $\tilde{\mathbf{z}}$ be an offer process that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains such that $\mathsf{c}(\tilde{\mathbf{z}}) \subseteq \mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$. Suppose that there exists a doctor $\bar{d} \in \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$, a hospital $\bar{h}$, and a contract $y \in (\mathsf{c}(\mathbf{x}) \cap X_{\bar{h}} \cap X_{\bar{d}}) \setminus \mathsf{c}(\tilde{\mathbf{z}})$ such that $\bar{d} \notin \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\tilde{\mathbf{z}})))$ and $y \succ_{\bar{d}} \emptyset$. Then there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{z}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$. If, in addition to the other requirements, $(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y\}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$, then $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})$.*

Proof of Claim 4. We first show how to construct a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{z}}$ provided that the conditions of Claim 4 are satisfied. Note that $\mathsf{c}(\tilde{\mathbf{z}}) \subseteq \mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$ and $\bar{d} \in \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}})))$ imply that $\bar{d} \neq \hat{d}$.

Let $\tilde{y}^1$ be the highest ranked contract in $(X_{\bar{d}} \cap X_{\bar{h}}) \setminus \mathsf{c}(\tilde{\mathbf{z}})$ with respect to $\succ_{\bar{d}}$. Clearly,

$\tilde{y}^1$ satisfies Condition 1 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$. Furthermore, given that $y \in \mathsf{c}(\mathbf{x})$, $\tilde{y}^1 \,\hat{\succeq}_{\bar{d}}\, y$, and that $\mathbf{x}$ is compatible with $\succ$, it has to be the case that $\tilde{y}^1 \in \mathsf{c}(\mathbf{x})$. Proceeding inductively, suppose that we have defined a sequence of $n \geq 1$ distinct contracts $\tilde{y}^1, \ldots, \tilde{y}^n \in \mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup X_{\hat{d}})$ such that $(\tilde{y}^1, \ldots, \tilde{y}^n)$ satisfies Conditions 1 and 2 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$. We will show that either $\bar{d} \in \mathsf{d}(R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}))$, so that $(\tilde{y}^1, \ldots, \tilde{y}^n)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$, or that there exists a contract $\tilde{y}^{n+1} \in \mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup X_{\hat{d}})$ that satisfies Condition 2 of the definition of a pre-run rejection chain at $\tilde{\mathbf{z}}$.

We claim that $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}) \neq \varnothing$. Since $\mathbf{x}$ ends with the rejection of $\hat{x}$, $|C^h(\mathsf{c}(\mathbf{x}))| \leq |C^h(\mathsf{c}(\check{\mathbf{x}}))|$ for all $h \in H$. To see this, note first that by Proposition 1 we can think of $\mathbf{x}$ as a combined offer process $\mathbf{x} \equiv (\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$, where $\hat{z}^1$ is the highest ranked acceptable contract in $X_{\hat{d}}$ with respect to $\succ_{\hat{d}}$ and, for all $m \in \{2, \ldots, M\}$, $\mathsf{d}(\hat{z}^m) \in \mathsf{d}(R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-2}\}))$ and $\hat{z}^m$ is the highest ranked contract in $X_{\mathsf{d}(z^m)} \smallsetminus (\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})$ with respect to $\succ_{\mathsf{d}(\hat{z}^m)}$. Now by observable substitutability we have that, for all m, $C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \subseteq C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) \cup \{\hat{z}^m\}$. In particular, $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})| + 1$. If there were an m such that $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| = |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})| + 1$, we would have that $R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\}) \smallsetminus R^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\}) = \varnothing$. However, this implies a contradiction to the observation that $\mathbf{x}$ ends with the rejection of contract $\hat{x}$.[35] Hence, we must have $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^m\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^{m-1}\})|$ for all m. Hence, we must have $|C^H(\mathsf{c}(\check{\mathbf{x}}) \cup \{\hat{z}^1, \ldots, \hat{z}^M\})| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}))|$. Therefore, since $\mathbf{x}$ is the combined offer process $\mathbf{x} \equiv (\check{\mathbf{x}}, \hat{z}^1, \ldots, \hat{z}^M)$, we must have that $|C^H(\mathsf{c}(\mathbf{x}))| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}))|$. By observable size monotonicity, we must have, for all $h \in H$, $|C^h(\mathsf{c}(\mathbf{x}))| \geq |C^h(\mathsf{c}(\check{\mathbf{x}}))|$. Together with $|C^H(\mathsf{c}(\mathbf{x}))| \leq |C^H(\mathsf{c}(\check{\mathbf{x}}))|$, this implies $|C^h(\mathsf{c}(\mathbf{x}))| = |C^h(\mathsf{c}(\check{\mathbf{x}}))|$.

Next, note that $\check{\mathbf{x}}$, $(\tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n)$, and $\mathbf{x}$ are all weakly observable and weakly compatible with $\succ$. Hence, $(\check{\mathbf{x}}, \tilde{\mathbf{z}}, \tilde{y}^1, \ldots, \tilde{y}^n, \mathbf{x})$ is weakly observable by Lemma 1. Since we also have that

[35] See Footnote 34.

$\mathsf{c}(\check{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\mathbf{x})$, observable size monotonicity implies that $|C^h(\mathsf{c}(\mathbf{x}))| \geq |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})| \geq |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})| \geq |C^h(\mathsf{c}(\check{\mathbf{x}}))|$ for all $h \in H$. Since $|C^h(\mathsf{c}(\mathbf{x}))| = |C^h(\mathsf{c}(\check{\mathbf{x}}))|$, we must have $|C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})| = |C^h(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})|$ and thus $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}) \neq \varnothing$. Furthermore, given that observable size monotonicity implies that $|R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})| \leq 1$, there has to be a unique contract $\bar{y}^{n+1} \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})$.

If $\mathsf{d}(\bar{y}^{n+1}) = \bar{d}$, we are done since $(\tilde{y}^1, \ldots, \tilde{y}^n)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$. If not, let $d^{n+1} \equiv \mathsf{d}(\bar{y}^{n+1})$. Since $\bar{y}^{n+1} \in R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\})$ and $\mathsf{c}(\check{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{n-1}\}$, observable substitutability implies $\bar{y}^{n+1} \notin R^H(\mathsf{c}(\check{\mathbf{x}}))$. Note that subsequent to $\check{\mathbf{x}}$, $\mathbf{x}$ ends as soon as a contract is rejected such that the associated doctor has already proposed all acceptable contracts.Since $\mathbf{x}$ ends with the rejection of $\hat{x}$ and since $d^{n+1} \neq \hat{d}$, this implies that there must be a contract in $\mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$ that is acceptable to d^{n+1}. Hence, we can let $\tilde{y}^{n+1}$ be the favorite contract of d^{n+1} in $\mathsf{c}(\mathbf{x}) \smallsetminus (\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$ and proceed.

Since the set of doctors is finite, there must exist a smallest integer $N \geq 1$ such that $\bar{d} \in \mathsf{d}(R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^N\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^{N-1}\}))$ and $\tilde{\mathbf{y}} = (\tilde{y}^1, \ldots, \tilde{y}^N)$ is a pre-run rejection chain at $\tilde{\mathbf{z}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$.

To complete the proof of Claim 4, we now establish that $(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{y\}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$ implies $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})$. We will prove by induction on n that $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$. For $n = 1$, $y \in \mathsf{c}(\hat{\mathbf{x}})$, $\tilde{y}^1 \mathrel{\hat{\succeq}_{\bar{d}}} y$, and the compatibility of $\hat{\mathbf{x}}$ with $\hat{\succ}$, imply that $\tilde{y}^1 \in \mathsf{c}(\hat{\mathbf{x}})$. Now assume that, for some $n < N$, we had already shown that $\{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$. Since $\tilde{\mathbf{z}}, (\tilde{y}^1, \ldots, \tilde{y}^n), \hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succ}$, Lemma 1 implies that $(\tilde{\mathbf{z}}, (\tilde{y}^1, \ldots, \tilde{y}^n), \hat{\mathbf{x}})$ is weakly observable. Since there are no observable violations of substitutes, we must have $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup \mathsf{c}(\hat{\mathbf{x}}))$. Since $\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \subseteq \mathsf{c}(\hat{\mathbf{x}})$ by the inductive assumption, we must have $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\} \cup \mathsf{c}(\hat{\mathbf{x}})) = R^H(\mathsf{c}(\hat{\mathbf{x}}))$. By the construction of $\tilde{\mathbf{y}}$, we must have that $\mathsf{d}(\tilde{y}^{n+1}) \notin \mathsf{d}(C^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}))$ and that $\tilde{y}^{n+1}$ is the highest ranked contract in $X_{\mathsf{d}(\tilde{y}^{n+1})} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\})$. Since $R^H(\mathsf{c}(\tilde{\mathbf{z}}) \cup \{\tilde{y}^1, \ldots, \tilde{y}^n\}) \subseteq$

$R^H(\mathsf{c}(\hat{\mathbf{x}}))$ and since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$, we must have $\tilde{y}^{n+1} \in \mathsf{c}(\hat{\mathbf{x}})$. This completes the proof of Claim 4. $\square$

With the help of the just established Claim 4, we now finish our proof of Claim 3. It will prove useful to introduce some additional notation and terminology. Let $\tilde{D} \subseteq D \smallsetminus \{\hat{d}\}$ be the set of all doctors $d \neq \hat{d}$ for whom $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \neq \varnothing$. Note that for any $d \in D \smallsetminus (\tilde{D} \cup \{\hat{d}\})$, we must have $[\mathsf{c}(\mathbf{x})]_d \subseteq [\mathsf{c}(\hat{\mathbf{x}})]_d$ given that $\hat{\mathbf{x}}$ is a complete offer process for $\hat{\succ}$. In particular, for any $d \in D \smallsetminus (\tilde{D} \cup \{\hat{d}\})$, we must have $(\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})) \cap X_d = \varnothing$ given that $\mathsf{c}(\hat{\mathbf{x}}') \subseteq \mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})$. Finally, let $\tilde{\mathbf{x}}$ be an offer process such that

1. $\tilde{\mathbf{x}}$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains,
2. $\mathsf{c}(\check{\mathbf{x}}) \subseteq \mathsf{c}(\tilde{\mathbf{x}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$, and
3. there is no other offer process $\mathbf{w}$ that can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains such that $\mathsf{c}(\tilde{\mathbf{x}}) \subsetneq \mathsf{c}(\mathbf{w}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$.

Note that since $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq X \smallsetminus X_{\hat{d}}$ and $\succ_{-\hat{d}} = \hat{\succ}_{-\hat{d}}$, it does not matter whether we use $\succ$ or $\hat{\succ}$ as the basis for pre-running rejection chains to construct $\tilde{\mathbf{x}}$. Note also that an offer process such as $\tilde{\mathbf{x}}$ must exist given that the set of contracts is finite. In the remainder of the proof we will establish that there exists a generalized pre-run rejection chain $\mathbf{z}^*$ at $\tilde{\mathbf{x}}$ such that the combined offer process $\mathbf{y}^* \equiv (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies all four properties of Claim 3.

Assume that we have already constructed a generalized pre-run rejection chain $\mathbf{z}$ at $\tilde{\mathbf{x}}$ that satisfies the following five properties:

(P1) $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \neq \varnothing$;

(P2) $\mathsf{c}(\mathbf{z}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$;

(P3) $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$;

(P4) For all $d \in \tilde{D}$, if $[\mathsf{c}(\hat{\mathbf{x}})]_d \nsubseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, then $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$;

(P5) For all $d \in D \setminus \{\hat{d}\}$, if $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d \neq \varnothing$, then $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_d$ contains the highest ranking contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ with respect to $\succ_d$.

Property 1 is satisfied when the construction of $\mathbf{z}$ is not complete—as long as $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ is not empty, we will be able to extend our generalized pre-run rejection chain. The second property ensures that $\mathbf{z}$ only includes contracts in $\mathbf{x}$. Property 3 requires that, as we build our pre-run rejection chain, any contract rejected during the combined offer process $(\hat{\mathbf{x}}, \mathbf{z})$ that is not rejected during the offer process $\hat{\mathbf{x}}$ is also rejected during the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$. Property 4 states that for each doctor employed after the offer process $\hat{\mathbf{x}}$, if there is some contract in $\hat{\mathbf{x}}$ with that doctor that is not rejected during the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z})$, then the contract that doctor obtains after $\hat{\mathbf{x}}$ is not rejected during the combined offer process $(\hat{\mathbf{x}}, \mathbf{z})$. Finally, the last property ensures that, for each doctor employed after the offer process $(\hat{\mathbf{x}}, \mathbf{z})$, that doctor obtains the highest ranked contract not yet rejected.

We show below how to extend a generalized pre-run rejection chain $\mathbf{z}$ that satisfies (P1) - (P5) into a strictly longer pre-run rejection chain that satisfies (P2) - (P5) and that contains at least one contract from $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ is finite, this implies the existence of a generalized pre-run rejection chain $\mathbf{z}^*$ at $\tilde{\mathbf{x}}$ that satisfies (P2) - (P5) and $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) = \varnothing$.

We will now argue that if a generalized pre-run rejection chain $\mathbf{z}^*$ at $\tilde{\mathbf{x}}$ satisfies (P2), (P3), and $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) = \varnothing$, then $\mathbf{y}^* \equiv (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies all four properties of Claim 3. Since $\tilde{\mathbf{x}}$ is obtained from $\check{\mathbf{x}}$ by pre-running rejection chains and since $\mathbf{z}^*$ is a generalized pre-run rejection chain at $\tilde{\mathbf{x}}$, the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z}^*)$ can be obtained from $\check{\mathbf{x}}$ by pre-running rejection chains, satisfying Condition 1 of Claim 3. Given that $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq X \setminus X_{\hat{d}}$ by the construction of $\tilde{\mathbf{x}}$ and given that $\mathsf{c}(\mathbf{z}^*) \subseteq X \setminus X_{\hat{d}}$ by (P2), we get that $\mathsf{c}((\tilde{\mathbf{x}}, \mathbf{z}^*)) \subseteq X \setminus X_{\hat{d}}$, satisfying Condition 2 of Claim 3. Since $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) = \varnothing$, we must have that $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{z}^*) \subseteq \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*) = \mathsf{c}(\mathbf{y}^*)$, so that $(\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies Condition 3 of Claim 3. Finally, since $\mathbf{z}^*$ satisfies (P3), we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}^*)) =$

$R^H(\mathsf{c}(\mathbf{y}^*))$, which implies that $\mathbf{y}^* = (\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies Condition 4 of Claim 3.

Note that properties (P4) and (P5) are not needed to establish that $(\tilde{\mathbf{x}}, \mathbf{z}^*)$ satisfies all four properties of Claim 3. However, (P4) and (P5) are essential in guaranteeing that we can extend a generalized pre-run rejection chain $\mathbf{z}$ that satisfies (P1) into a strictly longer pre-run rejection chain that contains at least one element of $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Before proceeding, note that the proof that $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}'))$ is trivial when $\mathsf{c}(\hat{\mathbf{x}}') \subseteq \mathsf{c}(\hat{\mathbf{x}})$: By Proposition 1 we can think of the cumulative offer process at $\hat{\succ}'$ as resulting from an ordering $\vdash$ such that, for all $y \in \mathsf{c}(\hat{\mathbf{x}})$ and all $z \in X \setminus \mathsf{c}(\hat{\mathbf{x}})$, $y \vdash z$. This implies that $[\mathsf{c}(\hat{\mathbf{x}})]_{\hat{h}} \subseteq \mathsf{c}(\hat{\mathbf{x}}')$ given that $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$ and given that $\hat{\succ}' = \hat{\succ}^{X_{\hat{h}}}$. Hence, $\mathsf{c}(\hat{\mathbf{x}}') \subseteq \mathsf{c}(\hat{\mathbf{x}})$ implies that $[\mathsf{c}(\hat{\mathbf{x}})]_{\hat{h}} = \mathsf{c}(\hat{\mathbf{x}}')$. Furthermore $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}))$ implies $\hat{x} \in C^H(\mathsf{c}(\hat{\mathbf{x}}'))$, so that there would be nothing left to show. Henceforth, we will therefore assume that $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \neq \varnothing$.

To construct the desired generalized pre-run rejection chain $\mathbf{z}^*$, we start by showing that (P1) - (P5) are satisfied when $\mathsf{c}(\mathbf{z}) = \varnothing$. Our assumption that $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \neq \varnothing$ immediately implies (P1). Property 2 is immediate when $\mathsf{c}(\mathbf{z}) = \varnothing$. Next, note that when $\mathsf{c}(\mathbf{z}) = \varnothing$, $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) = \varnothing$; hence, Property 3 is satisfied. Moreover, Property 4 is also satisfied since $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d$ is nonempty (as $d \in \tilde{D}$) and $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \cap R^H(\mathsf{c}(\hat{\mathbf{x}})) = \varnothing$ by the definitions of C^H and R^H. Finally, if Property 5 was not satisfied, there would be a doctor $d \in D \setminus \{\hat{d}\}$ and a contract $\tilde{z} \in X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ such that $\tilde{z} \succ_d C^H(\mathsf{c}(\tilde{\mathbf{x}}))$. We can assume without loss of generality that $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ with respect to $\succ_d$. We must have $d \notin \mathsf{d}(C^{\mathsf{h}(\tilde{z})}(\mathsf{c}(\tilde{\mathbf{x}})))$: Otherwise, the contract in $[C^H(\mathsf{c}(\tilde{\mathbf{x}}))]_d$ would have been proposed before $\tilde{z}$ so that $\tilde{\mathbf{x}}$ would not be weakly compatible with $\succ_d$. Now note that $\tilde{\mathbf{x}}$ and $\mathbf{x}$ are both weakly observable and weakly compatible with $\succ$. Hence, $(\tilde{\mathbf{x}}, \mathbf{x})$ is weakly observable by Lemma 1. Observable substitutability implies that $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\mathbf{x})) = R^H(\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\tilde{\mathbf{x}}))$. Since $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ and $\mathbf{x}$ is compatible with $\succ$, we must have $\tilde{z} \in \mathsf{c}(\mathbf{x})$. A completely analogous argument shows that

$\tilde{z} \in \mathsf{c}(\hat{\mathbf{x}})$.[36] Since $\tilde{z} \in (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$ and $d \notin \mathsf{d}(C^{\mathsf{h}(\tilde{z})}(\mathsf{c}(\tilde{\mathbf{x}})))$, we obtain a contradiction to the definition of $\tilde{\mathbf{x}}$ given that Claim 4 implies that there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{x}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subsetneq (\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})) \setminus X_{\hat{d}}$.

Now that we know that Properties 1–5 are satisfied when $\mathsf{c}(\mathbf{z}) = \varnothing$, we will show how to extend $\mathbf{z}$ into a strictly longer generalized pre-run rejection chain at $\tilde{\mathbf{x}}$ that satisfies (P1)–(P5) and that contains at least one contract from $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Let y^1 be the contract in $\mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ that appears first in the sequence $\hat{\mathbf{x}}'$ and let $d^1 \equiv \mathsf{d}(y^1)$. Note that $d^1 \neq \hat{d}$ and $\mathsf{h}(y^1) = \hat{h}$ since $(\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}})) \subseteq (X_{\hat{h}} \setminus X_{\hat{d}})$. We will now show that there exists a generalized pre-run rejection $\mathbf{y} = (y^1, \ldots, y^N)$ at $(\tilde{\mathbf{x}}, \mathbf{z})$ such that $(\mathbf{z}, \mathbf{y})$ satisfies (P2) - (P5).

Step 1: If $\tilde{y} \in X_{\hat{h}} \cap X_{d^1}$ is such that $\tilde{y} \succ_{d^1} y^1$, then $\tilde{y} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

Suppose the contrary. Assume without loss of generality that $\tilde{y}$ is the highest ranking contract in $(X_{\hat{h}} \cap X_{d^1}) \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ with respect to $\succ_{d^1}$. Recall that $\mathsf{c}(\hat{\mathbf{x}}') \setminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{x})$ given that doctors only rank contracts in $\mathsf{c}(\mathbf{x}) \cup \mathsf{c}(\hat{\mathbf{x}})$ as acceptable under $\succ$ and $\hat{\succ}$. Hence, since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \setminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by assumption, we must have that $y^1 \in \mathsf{c}(\mathbf{x})$. Therefore, given that $\tilde{y} \succ_{d^1} y^1$, the compatibility of $\mathbf{x}$ with $\succ$ implies that we must have $\tilde{y} \in \mathsf{c}(\mathbf{x})$. For this step, it is useful to define $\hat{\mathbf{x}}''$ to be the offer process that is obtained from $\hat{\mathbf{x}}'$ by deleting y^1 and all contracts that are proposed after y^1. Note that since $\hat{\mathbf{x}}''$ is weakly compatible with $\hat{\succ}$ and since $\tilde{y} \mathrel{\hat{\succ}}_{d^1} y^1$,[37] we must have $\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}''))$.

We show first that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. There are two cases to consider:

Case 1: $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \{\tilde{y}\}$

In this case, we must clearly have $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. By the assumption that $\mathbf{z}$ satisfies (P3), we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since we have assumed that $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, we must have $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

[36] Now note that $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are both weakly observable and weakly compatible with $\hat{\succ}$. Hence, $(\tilde{\mathbf{x}}, \hat{\mathbf{x}})$ is weakly observable by Lemma 1. Observable substitutability implies that $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}})) = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}))$. Since $\tilde{z}$ is the highest ranked contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}))$ and $\hat{\mathbf{x}}$ is compatible with $\hat{\succ}$, we must have $\tilde{z} \in \mathsf{c}(\hat{\mathbf{x}})$.

[37] Remember that $d^1 \neq \hat{d}$ and that, for all $d \neq \hat{d}$, $\hat{\succ}_d = \succ_d$.

Case 2: $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} \neq \{\tilde{y}\}$

We will show first that we must have $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \varnothing$ in this case. Suppose to the contrary that there is some $\tilde{y}' \in [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}}$. Since d^1 prefers each contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ to each contract in $[\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_{d^1}$,[38] we must have $\tilde{y}' \mathrel{\hat{\succ}_{d^1}} y^1$ and therefore also $\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}''))$. Note that $\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z}$ are all weakly observable and weakly compatible with $\hat{\succ}$. Hence, $(\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z})$ is weakly observable by Lemma 1. Since $\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ by the construction of $\hat{\mathbf{x}}''$[39], the absence of observable violations of substitutes implies that $\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. If $\tilde{y} \mathrel{\hat{\succ}_{d^1}} \tilde{y}'$, we must have $\tilde{y}' \notin \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$ since d^1 could not have proposed $\tilde{y}'$ before $\tilde{y}$ was rejected and $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$; given that $\tilde{y}' \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$ and $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by (P3), we obtain $\tilde{y}' \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, contradicting $\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. If $\tilde{y}' \mathrel{\hat{\succ}_{d^1}} \tilde{y}$, we obtain $\tilde{y} \notin \mathsf{c}(\hat{\mathbf{x}})$ since d^1 could not have proposed $\tilde{y}$ before $\tilde{y}'$ was rejected. In particular, $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Since $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by (P3) and since $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by assumption, we obtain $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, contradicting $\{\tilde{y}, \tilde{y}'\} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Hence, the assumption that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} \neq \varnothing$ necessarily leads to a contradiction.

Now given that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\hat{h}} = \varnothing$ and d^1 is associated with contract $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$, there must be a hospital $\tilde{h} \neq \hat{h}$ such that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{d^1} \cap X_{\tilde{h}} \neq \varnothing$: Otherwise $\hat{\mathbf{x}}$ would not be a complete offer process with respect to $\hat{\succ}$.

Next, we show that $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$. Suppose the contrary. Note that since d^1

[38] Note that each $d \in D \smallsetminus \{\hat{d}\}$ prefers their contracts in $\mathsf{c}(\hat{\mathbf{x}})$ to their contracts in $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ under the preference profile $\succ_{-\hat{d}} = \hat{\succ}_{-\hat{d}}$, i.e., for each doctor $d \in D \smallsetminus \{\hat{d}\}$, for all $y \in [\mathsf{c}(\hat{\mathbf{x}})]_d$ and all $z \in [\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})]_d$, we have that $y \succ_d z$. We must have that $x \in \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$ implies $x \succ_{\mathsf{d}(x)} \emptyset$ and that $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$. Since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$ and since $\hat{\succ}_{-\hat{d}} = \succ_{-\hat{d}}$, $x \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ implies that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(x)} \neq \varnothing$ and that the contract in $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(x)}$ ranks higher than all contracts in $\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ with respect to $\succ_{\mathsf{d}(x)}$. Finally, since there are no observable violations of substitutes, $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(x)}$ must be the lowest ranking contract in $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(x)}$ with respect to $\succ_{\mathsf{d}(x)}$.

[39] Note that since y^1 is the contract in $\mathsf{c}(\hat{\mathbf{x}}') \smallsetminus (\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ that appears first in the sequence $\hat{\mathbf{x}}'$, we must have $\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$.

prefers all contracts in $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1}$ to all contracts in $[\mathsf{c}(\hat{\mathbf{x}}')]_{d^1} \smallsetminus \mathsf{c}(\hat{\mathbf{x}}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$[40] and since $y^1 \in \mathsf{c}(\hat{\mathbf{x}}') \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$, there is at least one contract in $\mathsf{c}(\mathbf{x})$ that d^1 likes strictly less than all contracts in $\mathsf{c}(\hat{\mathbf{x}})$. Since $\mathbf{x}$ is compatible with $\succ$, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq \mathsf{c}(\mathbf{x})$. Now if there is a contract $\tilde{z} \in (\mathsf{c}(\hat{\mathbf{x}}) \cap X_{\tilde{h}} \cap X_{d^1}) \smallsetminus \mathsf{c}(\tilde{\mathbf{x}})$, we obtain a contradiction to the definition of $\tilde{\mathbf{x}}$: $d^1 \in \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$ and the feasibility of $C^H(\mathsf{c}(\tilde{\mathbf{x}}))$ imply that $d^1 \notin \mathsf{d}(C^{\tilde{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$. Claim 4 then implies that there exists a pre-run rejection chain $\tilde{\mathbf{y}}$ at $\tilde{\mathbf{x}}$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq (\mathsf{c}(\mathbf{x}) \cap \mathsf{c}(\hat{\mathbf{x}})) \smallsetminus X_{\hat{d}}$. Hence, $(\mathsf{c}(\hat{\mathbf{x}}) \cap X_{\tilde{h}} \cap X_{d^1}) \smallsetminus \mathsf{c}(\tilde{\mathbf{x}}) = \varnothing$ if $d^1 \in \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$. Since $\tilde{\mathbf{x}}$ and $\hat{\mathbf{x}}$ are weakly observable and weakly compatible with $\hat{\succ}$, Lemma 1 implies that $(\tilde{\mathbf{x}}, \hat{\mathbf{x}})$ is weakly observable. Since there are no observable violations of substitutes, we must have $R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\hat{\mathbf{x}}))$. Since $\mathsf{c}(\tilde{\mathbf{x}}) \subseteq \mathsf{c}(\hat{\mathbf{x}})$, $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\hat{\mathbf{x}})) = R^H(\mathsf{c}(\hat{\mathbf{x}}))$. But since $(\mathsf{c}(\hat{\mathbf{x}}) \cap X_{\tilde{h}} \cap X_{d^1}) \subseteq \mathsf{c}(\tilde{\mathbf{x}})$ and $d^1 \in \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$, we must have $(\mathsf{c}(\hat{\mathbf{x}}) \cap X_{\tilde{h}} \cap X_{d^1}) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}))$. This contradicts the assumption that $d^1 \in \mathsf{d}(C^{\tilde{h}}(\mathsf{c}(\hat{\mathbf{x}})))$.

We can now show that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$ and $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, $\tilde{y} \in \mathsf{c}(\tilde{\mathbf{x}})$ would imply an observable violation of substitutability given that we would then have $\tilde{y} \in R^H(\mathsf{c}(\tilde{\mathbf{x}})) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$; hence, $\tilde{y} \notin \mathsf{c}(\tilde{\mathbf{x}})$. Furthermore, note that $\tilde{y} \in \mathsf{c}(\hat{\mathbf{x}}) \cap \mathsf{c}(\mathbf{x})$ would yield another contradiction to the definition of $\tilde{\mathbf{x}}$ given that $\mathsf{h}(\tilde{y}) = \hat{h}$, $d^1 \notin \mathsf{d}(C^{\hat{h}}(\mathsf{c}(\tilde{\mathbf{x}})))$, and $\tilde{y} \notin \mathsf{c}(\tilde{\mathbf{x}})$. Hence, we must have $\tilde{y} \notin \mathsf{c}(\hat{\mathbf{x}})$ and $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Given that $\tilde{y} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ by (P3), we obtain that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$.

We will now complete the proof of Step 1 by showing that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \cap R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ necessarily leads to a contradiction. Since $\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z}$ are all weakly observably and weakly compatible with $\hat{\succ}$, $(\hat{\mathbf{x}}'', \hat{\mathbf{x}}, \mathbf{z})$ is weakly observable by Lemma 1. Hence, $R^H(\mathsf{c}(\hat{\mathbf{x}}'')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}'') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ given that there are no observable violations

[40]See Footnote 38 for an explanation.

of substitutes. Since $\mathsf{c}(\hat{\mathbf{x}}'') \subseteq \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})$, $R^H(\mathsf{c}(\hat{\mathbf{x}}'') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Combining this with $R^H(\mathsf{c}(\hat{\mathbf{x}}'')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}'') \cup \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, we obtain $R^H(\mathsf{c}(\hat{\mathbf{x}}'')) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. As explained in the first paragraph of the proof, the construction of $\hat{\mathbf{x}}''$ implies $\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}''))$, and so $\tilde{y} \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. But we have established previously that $\tilde{y} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and hence obtain a contradiction.

Step 2: Extending the generalized pre-run rejection chain while preserving (P5).

By Step 1, we can start a new generalized pre-run rejection chain at $(\tilde{\mathbf{x}}, \mathbf{z})$ with y^1. By Claim 4, there exist $N_1 - 1$ contracts $y^2, \ldots, y^{N_1}$ such that $\mathbf{y}^1 \equiv (y^1, \ldots, y^{N_1})$ is a pre-run rejection chain at $(\tilde{\mathbf{x}}, \mathbf{z})$ and $\mathsf{c}(\mathbf{y}^1) \subseteq \mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$. Note that for all $d \in D \setminus \{d^1, \hat{d}\}$ such that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_d \neq \varnothing$, the assumption that $\mathbf{z}$ satisfies (P5) and the fact that $\mathbf{y}^1$ is a pre-run rejection chain imply that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_d$ contains the highest ranking acceptable contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ with respect to $\succ_d$. Note also that for all $n \in \{2, \ldots, N_1\}$, y^n is $\mathsf{d}(y^n)$'s highest ranking acceptable contract in $X_d \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$.

If $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))]_{d^1}$ also contains the highest ranking acceptable contract in $X_{d^1} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ with respect to $\succ_{d^1}$, we can set $\mathbf{y} \equiv \mathbf{y}^1$ to obtain a new generalized pre-run rejection chain $\mathbf{z}' \equiv (\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ that satisfies (P5). If not, (P5) applied to $\mathbf{z}$ implies that $y^1 \in C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$. Let y^{N_1+1} be the highest ranking contract in $X_{d^1} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$ with respect to $\succ_{d^1}$. Note that we must have $\mathsf{h}(y^{N_1+1}) \neq \hat{h}$ since $y^{N_1+1} \succ_{d^1} y^1$ and y^1 is the highest ranking contract in $X_{\hat{h}} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}^1))$. Hence, we can start a new pre-run rejection at $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}^1)$ with y^{N_1+1}. Note that Claim 4 and $\mathsf{d}(y^{N_1+1}) = d^1 \neq \hat{d}$ jointly imply that the pre-run rejection chain that starts with y^{N_1+1} would only consist of contracts in $\mathsf{c}(\mathbf{x}) \setminus X_{\hat{d}}$. Proceeding in this fashion, we must eventually reach an integer N such that $(\mathbf{z}, y^1, \ldots, y^N)$ is a generalized pre-run rejection at $\tilde{\mathbf{x}}$ that contains y^1 and satisfies (P5). Hence, we can set $\mathbf{y} \equiv (y^1, \ldots, y^N)$ to obtain a new generalized pre-run rejection chain

$(\mathbf{z}, \mathbf{y})$ at $\tilde{\mathbf{x}}$ that satisfies (P5).

Step 3: The extended generalized pre-run rejection chain satisfies (P2) - (P4).

Let $\mathbf{y} = (y^1, \ldots, y^N)$ be the generalized pre-run rejection at $(\tilde{\mathbf{x}}, \mathbf{z})$ constructed in Step 2. It follows immediately from the construction in Step 2 that $\mathsf{c}(\mathbf{y}) \subseteq \mathsf{c}(\mathbf{x}) \smallsetminus X_{\hat{d}}$. Hence, $(\mathbf{z}, \mathbf{y})$ satisfies Property 2.

Now, define the offer process $\mathbf{w}$ that lists the contracts in $\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$ in order of appearance in $\mathbf{y}$ as follows: Set $w^1 \equiv y^1$ and $n_1 \equiv 1$. Now assuming that $w^1, \ldots, w^o$ and $n_1, \ldots, n_o$ have already been defined, set $w^{o+1} \equiv y^{n_{o+1}}$, where $n_{o+1} \equiv \min\{n \in \{1, \ldots, N\} : y^n \notin \{w^1, \ldots, w^o\}$ and $y^n \in \mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})\}$. Let O be such that $\{w^1, \ldots, w^O\} = \mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$. Note that the offer process $\mathbf{w} = (w^1, \ldots, w^O)$ will not in general be observable.

Next, we will show that, for all $o \in O$, $|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1$. Note first that, by the construction of $\mathbf{w}$, we must have that, for all $o \leq O$, $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o}\}$. Since $(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is observable, we must thus have that $(\hat{\mathbf{x}}, \mathbf{z}, \mathbf{w})$ is observable (even though $\mathbf{w}$ need not be observable). By observable size monotonicity, we must have, for all $o \leq O$, $|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1$.

In the next step of our proof, we will establish that, for all $o \in \{1, \ldots, O\}$, there exists a contract $\hat{w}^o$ with the following three attributes:

(A1) $\mathsf{d}(\hat{w}^o) = \mathsf{d}(w^{o+1})$, where we set $O + 1 \equiv 1$,

(A2) $\{\hat{w}^o\} = R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$,

(A3) $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$, where we set $N_{O+1} \equiv N + 1$.

Suppose that, for some $o \in \{1, \ldots, O\}$, the statement has been established for all $o' \in \{1, \ldots, o-1\}$.[41] Let $\tilde{w}$ be the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \smallsetminus$

[41]Note that, when $o = 1$, this assumption is vacuously satisfied as $\{1, \ldots, o-1\} = \varnothing$.

$R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ and note that since $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$, where we set $y^{n_{O+1}} \equiv y^1$.

There are two cases:

Case 1: Suppose that $\tilde{w} \in \mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$.

We first show that $\tilde{w} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$. By the inductive assumption, for each $o' \in \{1, \ldots, o-1\}$, we have by (A2) that $\hat{w}^{o'}$ is the unique contract in $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o'}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o'-1}\})$. Hence, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \ldots, \hat{w}^{o-1}\}$. By the inductive assumption, for each $o' \in \{1, \ldots, o-1\}$, we have by (A3) that $\hat{w}^{o'} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o'+1}-1}\})$. Combining this with the previously established fact that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \ldots, \hat{w}^{o-1}\}$, we obtain $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o-1}\})$. Since $n_{o+1} \geq n_o + 1$ and since $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ by the construction of $\tilde{w}$, we obtain that $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_o-1}\})$ and therefore also $\tilde{w} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Next, note that, by (P3) applied to $\mathbf{z}$, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$, $n_{o+1} - 2 \geq 0$, and since $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$ is observable, the absence of observable violations of substitutes implies that $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and therefore also $\tilde{w} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}))$. Finally, we must have $\tilde{w} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}))$ given that $\tilde{w} \notin \mathsf{c}(\hat{\mathbf{x}})$. Since $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) = [R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))] \cup [R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}))] \cup R^H(\mathsf{c}(\hat{\mathbf{x}}))$, we obtain the desired statement that $\tilde{w} \notin R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$.

Next, we show that $\tilde{w} \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$. Note that $\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succ}$,[42] so that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}})$ is weakly observable by Lemma 1. Since there are no observable violations of

[42]Remember that $\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}) \subseteq X \smallsetminus X_{\hat{d}}$.

substitutes, we must have $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$. By the construction of $\mathbf{w}$, we must have $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}$.[43] Since $\tilde{w} \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$, we must thus have $\tilde{w} \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$.

Hence, we can let $\hat{w}^o \equiv \tilde{w}$ to obtain a contract in $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$. Since $\hat{w}^o = \tilde{w}$ is the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ and since $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$, so that (A1) is satisfied. Next, given that we have established above that $|R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})| \leq 1$, we must have $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) = \{\hat{w}^o\}$, so that (A2) is satisfied. Finally, (A3) is satisfied since $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$.

Case 2: Suppose that $\tilde{w} \in \mathsf{c}(\hat{\mathbf{x}})$.

Throughout the proof of Case 2, keep in mind that, since $\tilde{w}$ is the unique contract in $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ and $\mathbf{y}$ is a generalized pre-run rejection chain, we must have $\mathsf{d}(\tilde{w}) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$.

We start by establishing that $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(\tilde{w})} = [\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$. There are two subcases to consider:

Subcase 1: $o < O$.

In this case, the construction of $\mathbf{y}$ and the assumption that $\mathbf{z}$ satisfies (P5) ensure that $y^{n_{o+1}}$ is the highest ranking contract in $X_{\mathsf{d}(y^{n_{o+1}})} \smallsetminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$. Given that all doctors $d \in D \smallsetminus \{\hat{d}\}$ prefer all contracts in $\mathsf{c}(\hat{\mathbf{x}})$ to all contracts in $\mathsf{c}(\mathbf{x}) \smallsetminus \mathsf{c}(\hat{\mathbf{x}})$,[44] and that $w^{o+1} = y^{n_{o+1}} \notin \mathsf{c}(\hat{\mathbf{x}})$, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$.

[43]Recall that $\tilde{\mathbf{x}} \subseteq \hat{\mathbf{x}}$ by construction.
[44]See Footnote 38 for an explanation.

Subcase 2: $o = O$, where $\mathsf{d}(\tilde{w}) = d^1$.

The assumption that $\mathbf{z}$ satisfies (P5) implies that $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))]_{d^1}$ contains the highest ranking contract in $X_{d^1} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ with respect to $\succ_{d^1}$. If there was an $n \leq N$ such that $y^1 \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\}) \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$, the construction of $\mathbf{y}$ in Step 2 would have therefore ensured that $n = N$. But in this case, we would have $y^1 = \tilde{w}$ and would hence obtain a contradiction to the assumption that $\tilde{w} \in \mathsf{c}(\hat{\mathbf{x}})$ as $y^1 \notin \mathsf{c}(\hat{\mathbf{x}})$. Hence, we must have $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))]_{d^1} = \{y^1\}$. But at the end of the combined offer process $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y})$, $[C^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))]_{d^1}$ has to contain the highest ranking contract in $X_{d^1} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ with respect to $\succ_{d^1}$ by (P5) applied to $(\mathbf{z}, \mathbf{y})$. Since d^1 ranks all contracts in $\mathsf{c}(\hat{\mathbf{x}})$ higher than the contracts in $\mathsf{c}(\mathbf{x}) \setminus \mathsf{c}(\hat{\mathbf{x}})$,[45] this implies $[\mathsf{c}(\hat{\mathbf{x}})]_{d^1} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$.

Second, we will establish that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w})} = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$. Note first that since $\hat{\mathbf{x}}$ is a complete offer process with respect to $\hat{\succ}$, for all $d \in D \setminus \{\hat{d}\}$ such that $[\mathsf{c}(\mathbf{x}) \setminus \mathsf{c}(\hat{\mathbf{x}})]_d \neq \varnothing$, we have that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \neq \varnothing$. Since $\mathsf{d}(\tilde{w}) = \mathsf{d}(w^{o+1})$ is associated with the contract $w^{o+1} \in \mathsf{c}(\mathbf{y}) \setminus \mathsf{c}(\hat{\mathbf{x}})$ by the construction of $\tilde{w}$, we must thus have $\mathsf{d}(\tilde{w}) \in \tilde{D} \setminus \{\hat{d}\}$. Since $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$, we must have $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ given that there are no observable violations of substitutes. Now $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$ and (P4) applied to $\mathbf{z}$ imply that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Next, note that $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-2}\})$ and the absence of observable violations of substitutes also imply that, for all $n \leq n_{o+1} - 2$, $\tilde{w} \notin R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^n\})$. Since, for all $n \geq 2$, y^n is the most preferred contract in $X_{\mathsf{d}(y^n)} \setminus R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n-1}\})$ with respect to $\succ_{\mathsf{d}(y^n)}$ and since all doctors prefer all contracts in $\mathsf{c}(\hat{\mathbf{x}})$ to all contracts in $\mathsf{c}(\mathbf{x}) \setminus \mathsf{c}(\hat{\mathbf{x}})$,[46] we must have $\mathsf{d}(y^{n_{o+1}}) = \mathsf{d}(\tilde{w}) \notin \mathsf{d}(\{w^2, \ldots, w^o\})$. By the inductive assumption that, for all $o' \in \{1, \ldots, O-1\}$, $\mathsf{d}(\hat{w}^{o'}) = \mathsf{d}(w^{o'+1})$, we

[45] See Footnote 38 for an explanation.
[46] See Footnote 38 for an explanation.

must thus have $\mathsf{d}(y^{n_{o+1}}) \notin \mathsf{d}(\{\hat{w}^1, \ldots, \hat{w}^{o-1}\})$. By the inductive assumption that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \ldots, \hat{w}^{o-1}\}$, we thus obtain that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq X \smallsetminus X_{\mathsf{d}(y^{n_{o+1}})}$. Given that we have already established that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$, we obtain that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \nsubseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$.

Third, we will show that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w})} = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$. As we have established above, we must have $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$. Since $\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}}$ are all weakly observable and weakly compatible with $\hat{\succ}$, Lemma 1 implies that $(\tilde{\mathbf{x}}, \mathbf{z}, \mathbf{y}, \hat{\mathbf{x}})$ is weakly observable. Since there are no observable violations of substitutes, $R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\}) \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$ and thus $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$. Finally, $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}$ by the construction of $\mathbf{w}$; hence, we must also have $[\mathsf{c}(\hat{\mathbf{x}})]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$. In particular, we obtain that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})} \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\})$.

Now let $\hat{w}^o$ be the unique contract in $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(\tilde{w})} = [C^H(\mathsf{c}(\hat{\mathbf{x}}))]_{\mathsf{d}(y^{n_{o+1}})}$. By the three statements we have already established above, we obtain that $\hat{w}^o \in R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^o\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^{o-1}\})$, so that (A2) is satisfied, and $\hat{w}^o \in R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^{n_{o+1}-1}\})$, so that (A3) is satisfied. Finally, (A1) is satisfied as $\mathsf{d}(\tilde{w}) = \mathsf{d}(w^{o+1}) = \mathsf{d}(y^{n_{o+1}})$. This completes the proof in Case 2.

We will now argue how Attributes (A1)–(A3) of the offer process $\mathbf{w}$ imply that the extended generalized pre-run rejection $(\mathbf{y}, \mathbf{z})$ satisfies (P3). Note first that (A2) and (A3) of $\mathbf{w}$ imply that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$. By the assumption that $\mathbf{z}$ satisfies (P3), we then obtain that $R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\}) \smallsetminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\})$. By the construction of $\mathbf{w}$, $\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{w^1, \ldots, w^O\} = \mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \ldots, y^N\}$; therefore,

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^N\}) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}})) \subseteq R^H(\mathsf{c}(\tilde{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \{y^1, \dots, y^N\}).$$

Finally, we will show that $(\mathbf{z}, \mathbf{y})$ also satisfies (P4). Note that, for all $d \in \mathsf{d}(\{w^1, \dots, w^O\})$, we have $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$. On the other hand, for any given $d \in D \setminus (\mathsf{d}(\{w^1, \dots, w^O\}) \cup \{\hat{d}\})$, we have

$$R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y})) \setminus R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z})) = \{\hat{w}^1, \dots, \hat{w}^O\} \subseteq X \setminus X_d.$$

In particular, for any $d \in \tilde{D} \setminus (\mathsf{d}(\{w^1, \dots, w^O\}) \cup \{\hat{d}\})$, we have that $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}) \cup \mathsf{c}(\mathbf{y}))$ only when $[C^H(\mathsf{c}(\hat{\mathbf{x}}))]_d \subseteq R^H(\mathsf{c}(\hat{\mathbf{x}}) \cup \mathsf{c}(\mathbf{z}))$. Since (P4) holds for $\mathbf{z}$, this establishes that $(\mathbf{z}, \mathbf{y})$ also satisfies (P4).

This completes the proof of Claim 3. □

As explained in the discussion after the statement of Claim 3, Claim 3 implies Claim 2. This completes the proof of Claim 2. □

A.6 Proof of Proposition 5

Fix a preference profile $\succ$. Let $\vdash$ be one ordering and $\mathbf{x} = (x^1, \dots, x^M)$ be the corresponding complete offer process, and let $\vdash'$ be another ordering and $\mathbf{y} = (y^1, \dots, y^N)$ be the corresponding complete offer process.

We show first that $\mathsf{c}(\mathbf{x}) \setminus \mathsf{c}(\mathbf{y}) = \varnothing$. Suppose by way of contradiction that $\mathsf{c}(\mathbf{x}) \setminus \mathsf{c}(\mathbf{y}) \neq \varnothing$ and let m be the smallest integer such that $x^m \notin \mathsf{c}(\mathbf{y})$. Let $\mathbf{x}' = (x^1, \dots, x^{m-1})$. Three facts follow immediately:

1. $\mathsf{d}(x^m) \notin \mathsf{d}(C^H(\mathsf{c}(x')))$, as $\mathbf{x}$ is an observable offer process.
2. $\mathsf{d}(x^m) \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{y})))$, as $x^m \succ_{\mathsf{d}(x^m)} \emptyset$, $x^m \notin \mathsf{c}(\mathbf{y})$, and $\mathbf{y}$ is a complete offer process.
3. $\mathsf{d}(x^m) \notin \mathsf{d}(\mathsf{c}(\mathbf{y}) \setminus \mathsf{c}(\mathbf{x}'))$, as $\mathsf{c}(\mathbf{y}) \cap X_{\mathsf{d}(x^m)} \subseteq \mathsf{c}(\mathbf{x}')$ since $x^m \notin \mathsf{c}(\mathbf{y})$, each $x \in X_{\mathsf{d}(x^m)}$ such that $x \succ_{\mathsf{d}(x^m)} x^m$ is in $\mathsf{c}(\mathbf{x}')$, and $\mathbf{y}$ is an offer process.

Now, since $\mathbf{x}'$ and $\mathbf{y}$ are both compatible with respect to the same preference profile $\succ$, we can apply Lemma 1 to infer that $(\mathbf{x}', \mathbf{y})$ is weakly observable. Since C^h is observably substitutable across doctors for all $h \in H$, we must have that, if $\mathsf{d}(x^m) \notin C^H(\mathsf{c}(\mathbf{x}'))$ and $\mathsf{d}(x^m) \notin \mathsf{d}(\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{x}'))$, then $\mathsf{d}(x^m) \notin C^H(\mathsf{c}(\mathbf{x}') \cup \mathsf{c}(\mathbf{y})) = C^H(\mathsf{c}(\mathbf{y}))$, where the last equality follows from the fact that $\mathsf{c}(\mathbf{x}') \subseteq \mathsf{c}(\mathbf{y})$ by construction. But this statement and the three facts we showed previously can not be true simultaneously; thus, we have a contradiction.

The proof that $\mathsf{c}(\mathbf{y}) \smallsetminus \mathsf{c}(\mathbf{x}) = \varnothing$ is analogous.

A.7 Proof of Theorem 5

Fix a profile of choice functions $C = (C^h)_{h \in H}$ that are observably substitutable across doctors, a preference profile $\succ = (\succ_d)_{d \in D}$ for the doctors, and an ordering $\vdash$ of the elements of X. For any $t \geq 1$, let y^t denote the (unique) contract that is offered in Step t of the cumulative offer process with respect to $\vdash$ and $\succ$ and set $A^t \equiv \{y^1, \ldots, y^t\}$.

We first show by induction on t that $C^H(A^t)$ is a feasible outcome. For $t = 0$, there is nothing to show. So suppose the statement is true up to some $t \geq 0$ and consider Step $t+1$. Let $h^{t+1} \equiv \mathsf{h}(y^{t+1})$. Note that for any $h \neq h^{t+1}$, we have that $A^t = A^{t+1}$ and $C^h(A^t) = C^h(A^{t+1})$. Now consider an arbitrary contract $x \in C^{h^{t+1}}(A^{t+1}) \smallsetminus \{y^{t+1}\}$. Note that if $x \in R^{h^{t+1}}(A^t)$, observable substitutability across doctors implies $\mathsf{d}(x) \in \mathsf{d}(C^{h^{t+1}}(A^t))$. Hence, $x \in C^{h^{t+1}}(A^{t+1}) \smallsetminus \{y^{t+1}\}$ and the inductive assumption imply that $\mathsf{d}(x) \notin \mathsf{d}(C^h(A^t)) = \mathsf{d}(C^h(A^{t+1}))$, for all $h \neq h^{t+1}$. This shows that $C^H(A^{t+1})$ is a feasible outcome.

Next, we will show that $A \equiv C^H(A^T)$ is stable. By construction, A is individually rational for hospitals. Moreover, each doctor only proposes acceptable contracts. To see that A is unblocked, consider an arbitrary set of contracts $Z \subseteq X \smallsetminus A$ such that $Z \succ_d A$ for all $d \in \mathsf{d}(Z)$. As every doctor proposes during the cumulative offer process every contract preferable to their assigned contract, we must have $Z \subseteq A^T \smallsetminus A$. Since $A = C^H(A^T)$ and $Z \subseteq X \smallsetminus A$, irrelevance of rejected contracts implies $A = C^H(A \cup Z)$.[47] Hence, Z is not a

[47] Example 6 in Appendix B.1 shows that the irrelevance of rejected contracts condition is necessary

blocking set of A.

A.8 Proof of Theorem 6

Let $h \in H$ be an arbitrary hospital and assume that C^h is not observably substitutable across doctors. Let $\mathbf{x} = (x^1, \ldots, x^M) \in X_h$ be an observable offer process for which there exists a contract $x \in \mathsf{c}(\mathbf{x})$ such that $x \in R^h(\{x^1, \ldots, x^{M-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^M\})$ even though $\mathsf{d}(x) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{M-1}\}))$. Assume without loss of generality that $\mathbf{x}$ is *minimal* in the sense that, for all observable offer processes $\mathbf{y} = (y^1, \ldots, y^N)$ such that $\mathsf{c}(\mathbf{y}) \subsetneq \mathsf{c}(\mathbf{x})$, $y \in R^h(\{y^1, \ldots, y^{N-1}\}) \smallsetminus R^h(\{y^1, \ldots, y^N\})$ implies $\mathsf{d}(y) \in \mathsf{d}(C^h(\{y^1, \ldots, y^{M-1}\}))$.

Let $\bar{x}$ be a contract between $\mathsf{d}(x)$ and a hospital $\bar{h} \neq h$ and $\bar{x}^M$ be a contract between $\mathsf{d}(x^M)$ and $\bar{h}$.

For the doctors, we define $\succ$ by setting

1. for all m, m' such that $m < m'$ and $\mathsf{d}(x^m) = \mathsf{d}(x^{m'})$, $x^m \succ_{\mathsf{d}(x^m)} x^{m'} \succ_{\mathsf{d}(x^m)} \emptyset$,
2. $\bar{x} \succ_{\mathsf{d}(x)} \emptyset$ and, for all $m \in \{1, \ldots, M-1\}$ such that $\mathsf{d}(x^m) = \mathsf{d}(x)$, $x^m \succ_{\mathsf{d}(x)} \bar{x}$, and
3. $\bar{x}^M \succ_{\mathsf{d}(x^M)} x^M$ and, for all $m \in \{1, \ldots, M-1\}$ such that $\mathsf{d}(x^m) = \mathsf{d}(x^M)$, $x^m \succ_{\mathsf{d}(x^M)} \bar{x}^M$.

For $\bar{h}$, we set

$$C^{\bar{h}}(Y) = \begin{cases} \{\bar{x}\} & \bar{x} \in Y \\ \{\bar{x}^M\} & \bar{x} \notin Y \text{ and } \bar{x}^M \in Y \\ \varnothing & \text{otherwise.} \end{cases}$$

We show first that for any ordering $\vdash$, the set of contracts proposed in the cumulative offer process with respect to $\succ$ and $\vdash$ must be $\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$. This will be sufficient to prove Theorem 6 since $C^H(\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}) = C^h(\{x^1, \ldots, x^M\}) \cup \{\bar{x}\}$ and $\mathsf{d}(\bar{x}) = \mathsf{d}(x) \in \mathsf{d}(C^h(\{x^1, \ldots, x^M\}))$, so that the outcome of any cumulative offer process for $\succ$, $C^H(\mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\})$, is not even feasible.

to guarantee the existence of stable outcomes even when the choice functions of hospitals are observably substitutable and observably size monotonic.

For the remainder, fix an arbitrary ordering $\vdash$ of the set of contracts and let $\mathbf{y}$ be the sequence of contracts that is produced by the cumulative offer process with respect to $\succ$ and $\vdash$. Note that we must have $\mathsf{c}(\mathbf{y}) \subseteq \mathsf{c}(\mathbf{x}) \cup \{\bar{x}, \bar{x}^M\}$ since doctors only rank contracts in the latter set as acceptable. Now suppose first that there is an m such that $x^m \notin \mathsf{c}(\mathbf{y})$. Without loss of generality, assume that $\{x^1, \ldots, x^{m-1}\} \subseteq \mathsf{c}(\mathbf{y})$. By the rules of cumulative offer processes, $\mathbf{y}$ must be a complete offer process with respect to $\succ$. Since $x^m \notin \mathsf{c}(\mathbf{y})$ and $x^m \succ_{\mathsf{d}(x^m)} \emptyset$, we must have $\mathsf{d}(x^m) \in \mathsf{d}(C^H(\mathsf{c}(\mathbf{y})))$. We will distinguish two cases:

1. $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$

   Since $x^m \notin \mathsf{c}(\mathbf{y})$, the minimality of $\mathbf{x}$ implies that, for all observable offer processes $\tilde{\mathbf{y}} = (\tilde{y}^1, \ldots, \tilde{y}^O)$ such that $\mathsf{c}(\tilde{\mathbf{y}}) \subseteq \mathsf{c}(\mathbf{y})$, $\tilde{y} \in R^h(\{\tilde{y}^1, \ldots, \tilde{y}^{O-1}\}) \smallsetminus R^h(\{\tilde{y}^1, \ldots, \tilde{y}^O\})$ only if $\mathsf{d}(\tilde{y}) \in \mathsf{d}(C^h(\{\tilde{y}^1, \ldots, \tilde{y}^{O-1}\}))$. Hence, Lemma 1 implies that $((x^1, \ldots, x^{m-1}), \mathbf{y})$ is weakly observable.[48] Now given that $x^m \notin \mathsf{c}(\mathbf{y})$, the compatibility of $\mathbf{y}$ with $\succ$ implies that $\{x^m, \ldots, x^M\}_{\mathsf{d}(x^m)} \cap \mathsf{c}(\mathbf{y}) = \varnothing$. Since $\mathbf{x}$ is observable, we must have $\mathsf{d}(x^m) \notin \mathsf{d}(C^h(\{x^1, \ldots, x^{m-1}\}))$. But given that $\{x^1, \ldots, x^{m-1}\} \subseteq \mathsf{c}(\mathbf{y})$ and $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$, there must exist an $n \leq N$ and a contract $y \in \{x^1, \ldots, x^{m-1}\}_{\mathsf{d}(x^m)}$ such that $y \in R^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^{n-1}\}) \smallsetminus R^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^n\})$ and $\mathsf{d}(y) \notin C^h(\{x^1, \ldots, x^{m-1}\} \cup \{y^1, \ldots, y^{n-1}\})$. This contradiction shows that $\mathsf{d}(x^m) \in \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$ is impossible.

2. $\mathsf{d}(x^m) \in \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\mathbf{y})))$

   By construction of $C^{\bar{h}}$ and $\succ$, we must have $\mathsf{d}(x^m) \in \{\mathsf{d}(\bar{x}), \mathsf{d}(\bar{x}^M)\}$. It is easy to see that $\mathbf{y}$ can only be a complete offer process with respect to $\succ$ when $m = M$ and $\bar{x}^M \in C^{\bar{h}}(\mathsf{c}(\mathbf{y}))$. But if $m = M$, we have that $\mathsf{c}(\mathbf{y})_h = \{x^1, \ldots, x^{M-1}\}$ and hence $\mathsf{d}(x) \notin \mathsf{d}(C^h(\mathsf{c}(\mathbf{y})))$. Since $\mathbf{y}$ is a complete offer process with respect to $\succ$, we must

[48] That we are able to use Lemma 1 follows since the cumulative offer process with respect to $\succ' \equiv \succ^{\mathsf{c}(\mathbf{y})}$ and $\vdash$ must also produce the offer process $\mathbf{y}$. Hence, we can restrict attention to an economy in which only contracts in $\mathsf{c}(\mathbf{y})$ are available. Since $\mathsf{c}(\mathbf{y}) \subsetneq \mathsf{c}(\mathbf{x})$, the minimality of $\mathbf{x}$ implies that the choice function of h is observably substitutable across doctors in this associated economy.

then have that $\bar{x} \in \mathsf{c}(\mathbf{y})$ and hence, $\bar{x}^M \notin C^{\bar{h}}(\mathsf{c}(\mathbf{y}))$. This contradiction shows that $\mathsf{d}(x^m) \in \mathsf{d}(C^{\bar{h}}(\mathsf{c}(\mathbf{y})))$ is impossible.

Now given that $\{x^1, \ldots, x^M\} \subseteq \mathsf{c}(\mathbf{y})$, the compatibility of $\mathbf{y}$ with $\succ$ implies that $\bar{x}^M \in \mathsf{c}(\mathbf{y})$ as $\bar{x}^M \succ_{\mathsf{d}(x^M)} x^M$. But then $\mathbf{y}$ is observable only if there is an n such that $\bar{x}^M \in R^{\bar{h}}(\{y^1, \ldots, y^n\})$. Since the last statement is only possible when $\bar{x} \in \{y^1, \ldots, y^n\} \subseteq \mathsf{c}(\mathbf{y})$, we must have $\mathsf{c}(\mathbf{y}) = \{x^1, \ldots, x^N\} \cup \{\bar{x}, \bar{x}^M\}$.

B Examples

B.1 Necessity of Irrelevance of Rejected Contracts

In this section, we present an example showing that the irrelevance of rejected contracts condition is necessary for the stability of the cumulative offer mechanism—even when choice functions are observably substitutable and observably size monotonic. Proposition 1 in Aygün and Sönmez (2012) establishes that substitutability and size monotonicity imply irrelevance of rejected contracts. The following example will show that if substitutability and/or size monotonicity are weakened to observable substitutability and/or observable size monotonicity, irrelevance of rejected contracts is crucial in order to ensure that the cumulative offer mechanism is stable.

Example 6. Consider a setting in which $H = \{h\}$, $D = \{d, e\}$, and $X = \{x, y, \hat{y}\}$, with $\mathsf{h}(x) = \mathsf{h}(y) = \mathsf{h}(\hat{y}) = h$, $\mathsf{d}(x) = d$, and $\mathsf{d}(y) = \mathsf{d}(\hat{y}) = e$. Suppose that the choice function of

h is as follows:

$$C^h(\{x\}) = \{x\}$$
$$C^h(\{y\}) = \{y\}$$
$$C^h(\{\hat{y}\}) = \{\hat{y}\}$$
$$C^h(\{x,y\}) = \{x\}$$
$$C^h(\{x,\hat{y}\}) = \{x,\hat{y}\}$$
$$C^h(\{y,\hat{y}\}) = \{\hat{y}\}$$
$$C^h(\{x,y,\hat{y}\}) = \{\hat{y}\}.$$

It is straightforward to verify that C^h is observably substitutable and observably size monotonic. Let $\succ$ be a preference profile that is consistent with $((x,y,\hat{y}),\{x,y,\hat{y}\})$.[49] For the ordering $\vdash$ such that $x \vdash y \vdash \hat{y}$, all contracts that are available in the economy are actually proposed. Since $C^h(\{x,y,\hat{y}\}) = \{\hat{y}\}$, we have that $[\mathcal{C}^{\vdash}(\succ)]_d = \varnothing$. But the outcome $\{\hat{y}\}$ is blocked by $\{x\}$.

B.2 Observably Substitutability Does Not Imply Bilateral Substitutability or Substitutable Completability

In this section, we present an example of an observably substitutable, observably size monotonic, and non-manipulatable choice function this is not bilaterally substitutable nor substitutably completable. We do this by, essentially, combining our Example 2 of a choice function which is observably substitutable, observably size monotonic, and non-manipulatable but not substitutable completable with the example in Appendix D of Hatfield and Kominers (2015) of a choice function that is substitutably completable (and, hence, observably substitutable, observably size monotonic, and non-manipulatable) but not bilaterally substitutable.

[49] Recall that the preferences $\succ$ *are consistent with* $(\mathbf{y}, Y)$ if $\succ$ is consistent with Y and $\mathbf{y}$ is compatible with $\succ$.

Example 7. Consider a setting in which $H = \{h\}, D = \{d, e, f\} \cup \{i, j, k\}$, and $X = \{x, y, z, \hat{x}, \hat{y}, \hat{z}\} \cup \{u, w, \hat{w}, v\}$, with

$$\begin{aligned}
h &= \mathsf{h}(x) = \mathsf{h}(y) = \mathsf{h}(z) = \mathsf{h}(\hat{x}) = \mathsf{h}(\hat{y}) = \mathsf{h}(\hat{z}) = \mathsf{h}(u) = \mathsf{h}(v) = \mathsf{h}(\hat{v}) = \mathsf{h}(w),\\
d &= \mathsf{d}(x) = \mathsf{d}(\hat{x}),\\
e &= \mathsf{d}(y) = \mathsf{d}(\hat{y}),\\
f &= \mathsf{d}(z) = \mathsf{d}(\hat{z}),\\
i &= \mathsf{d}(u),\\
j &= \mathsf{d}(v) = \mathsf{d}(\hat{v}),\\
k &= \mathsf{d}(w).
\end{aligned}$$

Similar to Example 2, let $\bar{C}^h$ be be induced by the preferences

$$\{\hat{x}, z\} \succ \{\hat{z}, x\} \succ \{\hat{z}, y\} \succ \{\hat{x}, y\} \succ \{x, y\} \succ \{z, y\} \succ \{\hat{x}, \hat{z}\} \succ \{x, z\} \succ \\ \succ \{y\} \succ \{\hat{z}\} \succ \{\hat{x}\} \succ \{x\} \succ \{z\} \succ \varnothing.$$

note that $\bar{C}^h$ is not substitutably completable but is observably substitutable, observably size monotonic, and non-manipulatable.

Similar to the example in Appendix D of Hatfield and Kominers (2015), let $\tilde{C}^h$ be induced by the preferences

$$\{u, v, w\} \succ \{\hat{v}\} \succ \{u, v\} \succ \{u, w\} \succ \{v, w\} \succ \{u\} \succ \{v\} \succ \{w\} \succ \varnothing;$$

note that $\tilde{C}^h$ is not bilaterally substitutable but is observably substitutable, observably size monotonic, and non-manipulatable.

Let $C^h(Y) \equiv \bar{C}^h(Y) \cup \tilde{C}^h(Y)$ for all $Y \subseteq X$. It follows immediately that C^h is observably substitutable, observably size monotonic, and non-manipulatable but not substitutably completable or bilaterally substitutable.