College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case

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1 Introduction

Affirmative action policies in societies with heterogeneous populations are increasingly popular and are often considered necessary for equalizing opportunities for certain demographic groups. The United States and Brazil are examples of countries with greatly heterogeneous populations in terms of wealth and racial backgrounds. One way to mitigate the problem of inequality between individuals who belong to different racial or gender groups or come from families with different income levels is through affirmative action. Affirmative action is a method of positive discrimination in favor of a certain groups of people to close socioeconomic gaps that exist between different groups as a result of historic discriminatory practices. This paper studies affirmative action in college admission in Brazil where the goal is to give underrepresented groups increased chances of attending better universities.

The Brazilian federal higher education system comprises of 59 universities and 38 institutes of education, science and technology, with an annual inflow of about one million students to its undergraduate programs. Following an increasing role for affirmative action for students of African descent and of low-income families in terms of access to public universities[1], the Brazilian congress enacted in August 2012 a law establishing the implementation of a series of affirmative action policies throughout said system.

The law established that 50% of the seats in each program offered in those institutions[2] should be used for the affirmative action policies. In order to claim the privilege of having higher priority in the access to those seats, a student must complete the three years of high-school in a public institution (being it local, state or federal). When assigning students to at least 50% of those seats, the university must also give higher priority to students who claim the privilege associated with being low-income (and give documentation proving such status as defined in law.) Additionally, when assigning a number of seats in the same proportion of the aggregate number of blacks, browns and indians (here referred to as "minorities") in the state in which the institution is, the university should give higher priority to students who claim the privilege associated with being a minority. We will throughout this chapter talk in terms of seats giving higher priority to students who claim some privileges, and denote those as "**public HS privilege**", "**low-income privilege**" and "**minority privilege**".

In a state where minorities constitute 25% of the population, for example, a program with capacity of 80 will have 40 seats giving higher priority for students claiming public HS privilege. At least 20 of those should give higher priority for those claiming low-income privilege, and 10 for those claiming minority privilege.

In October of the same year, Brazil's Ministry of Education published an ordinance specifying some details on the implementation of the affirmative action law as well as a suggested mechanism for choosing students while satisfying those policies. Starting in the student selection processes of 2013, based on our observations, those recommendations were widely adopted as the new selection criteria.

[1]For detailed information about history of affirmative action in Brazil, check Moehlecke (2003).

[2]In Brazil, like in the Turkish system studied in Balinski and Sonmez (1999), students apply directly to a specific program in the university, differently from other countries like the US where students simply apply to the university and once there chooses majors or programs to pursuit.

The key distinctive issue presented by the privileges proposed in the law is the fact that they are multidimensional. That is, students may belong to one or more of the groups specified. For instance, a low-income white student from public high school qualifies for the low-income privilege but not for the minority privilege. Although the literature for affirmative action from a mechanism design perspective has seen many important contributions, as in Abdulkadiroglu and Sönmez (2003), Westkamp (2013) and Hafalir et al. (2013), to the best of our knowledge none of them are able to respond to the challenge introduced by these types of privileges.

Another unique aspect of this case is that students are not obligated to apply to the universities using any of the privileges for affirmative action to groups to which they belong. This is due to the fact that being selected through the affirmative action policies is an "opt-in" procedure, that is, those students who are object of those privileges may choose not to be selected through that special criterion. Therefore, some students may choose to "hide" whether they belong to some of the three groups mentioned above, depending on the mechanism used for the assignments.

Starting in 2010, a new centralized system[3] was put in place to match students to federal universities. Although the study of the characteristics of that system is outside of the scope of this paper, the problems identified here are still present in it, and moreover it shows that there is a tendency for centralization of that process. Methods that could improve upon the current system in a centralized way (as the one that we present in this chapter,) may therefore have a direct application and impact.

The problem of allocating indivisible goods in the absence of money is studied in many papers, starting from the seminal paper by Gale and Shapley (1962). They study a college admissions market where students have preferences over colleges and colleges have preferences over sets of students to be admitted. The market clearing condition that they defined, stability, is still in use (sometimes with variations) and considered as one of the most important goals that mechanism designers consider for matching problems. They also introduce the celebrated student-proposing deferred acceptance algorithm (DA) to find a stable allocation. The DA mechanism is also utilized in many applied and theoretical papers in the matching literature. The centralized algorithm we suggest in this paper, the cumulative offer algorithm, is also a variation of the DA algorithm.

The school choice with affirmative action problem consists of two parts. The first part is the schools' criteria for choosing students, which we denote a *choice function*. A choice function provides a set of students that are selected for any possible set of students that apply for a given school. The second part is the algorithm that the central authority uses to allocate school seats to students using the schools' choice functions.

The first approach to this problem from a mechanism design perspective is the work of Abdulkadiroglu and Sönmez (2003). They analyze the system in Boston (denoted *Boston Mechanism*), which gave students higher priorities in schools in their neighborhoods or in schools in which students have a sibling already attending. By giving these priorities, the Boston Mechanism positively discriminates some students for certain schools. Abdulkadiroglu and Sönmez (2003) propose two algorithms, DA and top trading cycles (TTC), as alternatives for the

[3]The Unified System of Selection, denoted SISU.

Boston school choice algorithm, while keeping priorities of the schools as given. They show that the DA yield outcomes that are stable and efficient from the students' perspective. Also, DA is not manipulable, i.e. no student can manipulate their preferences and obtain a better school assignment. Subsequently, Abdulkadiroglu (2005) considers the college admission problem with affirmative action policy, and shows sufficient conditions on the schools' preferences to recover the properties of the DA algorithm.

In a recent paper, Westkamp (2013) studies the German university admission system in which reserved seats are transferred to different subpopulations in case of lack of applications. In this matching with complex constraints problem, the author specifies a method for schools to choose sets of students in any given case and designs a mechanism that gives a stable allocation under these circumstances. In another recent paper, Kamada and Kojima (2012) study the Japanese Residency Matching Program, where there are quotas for regions in order to help rural regions attract more residents. In the mechanism they study, the government sets a target capacity for each hospital to implement these quotas. They show that using target capacities may result in inefficiencies and that violating these targets may improve over the inefficiencies.

In 2012, Kojima showed that in affirmative action problems with two groups (majorities and minorities), using maximum quotas (that is, a maximum number of students for some types) for even one side may be inefficient and hurt all members of the minority group – the group which the policy intends to help. In a subsequent paper, Hafalir et al. (2013) study the school choice problems with affirmative action for minorities. They show the deficiencies of utilizing maximum quotas for school choice problems with affirmative action: welfare losses and wasted seats. Switching the system to DA with *minority reserves* instead solves the problem of wasted seats and signifficantly improves students' welfare.

Our model is built upon the matching with contracts model described by Hatfield and Milgrom (2005). Hatfield and Milgrom (2005) connect the matching problem of indivisible goods and the labor market model. They show that the foundations of a labor market model where workers can be hired by many alternative contracts (Kelso and Crawford, 1982) are also achievable in matching markets. This paper is very important because it not only subsumes and unifies these two problems but also relates the DA algorithm with fixed point techniques in lattice theory. In our problem, students do not have to declare their true demographic status through the privileges that they claim, i.e. a minority student can be admitted as a non-minority student. Hence, as in a matching with contracts problem, students can be admitted in different ways to schools.

The remaning of this chapter is structured as follows. In section 2 we present the mechanism suggested by the Ministry of Education and currently used by the universities surveyed. In section 3, we introduce the matching with contracts model that we apply to the school choice problem with affirmative action. In section 4, we introduce the Multidimensional Brazil Privileges Choice Function and we build upon the choice function defined to describe a mechanism – Student Optimal Stable Mechanism – that matches students to colleges in a centralized way, satisfies stability, is strategy-proof and fair. In section 5, we show that even for a single college, the currently used Brazil Reserves Choice Function induces a game with multiple Nash Equilibria in which strategically sophisticated students may obtain advantage by strategizing over the privileges that they claim. We also show that the current mechanism is not fair and cannot guarantee the satisfaction of the affirmative action objectives when they are feasible. In section 6, we conclude. All the proofs are given in the Appendix section.

2 Brazilian Reserves Choice Function

For the most part, until 2010, college admissions in Brazil worked essentially in a decentralized way. Students applied for a single program in each university that they desire to (Ex: History at University of Brasilia or Biology at Federal University of Minas Gerais). By using some combination of scores in a national exam and sometimes exams particular to those programs, the universities ranked them and accepted the top applicants to each program up to the programs' capacities, putting the remaining ones in waiting lists.

Among those accepted, typically some would not enroll because they were also accepted by other universities and courses of their preference. The universities would then proceed to a second round, accepting students from the waitlist following their ranking. Depending on the university this might be followed by third and fourth rounds.

The introduction of the reserves law has not changed the decentralized nature of the system yet. But the centralized online system used for some universities gives a strong signal that officials in charge of college admissions in Brazil are open to utilize a centralized method, which is shown in many papers to improve efficiency and reduce wasted seats in colleges. On the other hand, the affirmative action law changed the choice rules of universities in each step in an attempt to satisfy the affirmative action objectives. The rules used by the universities surveyed in this work are, essentially, strict implementations (or small variations) of the one suggested by Brazil's Ministry of Education. This rule tells the set of students to be chosen from any set of applicants and will be denoted as the class of Brazil Reserves Choice Function (BRCF). It suggests that the seats for each program should be split into five subsets. For any program with capacity Q, the five distinct subsets are:

- A set Q_{mi} with $\lceil \frac{Q}{4} r^m \rceil$ seats which give priority to students who claim public HS, minority and low-income privileges,
- A set Q_{Mi} with $\lceil \frac{Q}{4}(1 - r^m) \rceil$ seats which give priority to students who claim public HS and low-income privileges only,
- A set Q_{mI} with $\lceil \frac{Q}{4} r^m \rceil$ seats which give priority to students who claim public HS and minority privileges only,
- A set Q_{MI} with $\lceil \frac{Q}{4}(1 - r^m) \rceil$ seats which give priority to students who claim public HS privilege only,
- A set Q_{-} with the remaining seats.

where r^m is the ratio of minorities in the state where that program (college) is located.

Given the students who apply for each of those, the ones better ranked on the entrance exam are accepted up to the capacity of the set. If there are enough applicants for each of those sets, the affirmative action objectives, as described by the law, are satisfied. In case the number of students who apply for some of those sets is smaller than their capacity, those seats are filled following the priority structure below:

- If there are seats available in Q_{mi}, those are made available:
  - to students claiming low-income and public HS privileges only, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming public HS privileges only, then
  - to any student
- If there are seats available in Q_{Mi}, those are made available:
  - to students claiming low-income, minority and public HS privileges, then
  - to students claiming minority and public HS privileges only, then
  - to students claiming HS privilege only, then
  - to any student
- If there are seats available in Q_{mI}, those are made available:
  - to students claiming public HS privilege only, then
  - to students claiming low-income, minority and public HS privileges, then
  - to students claiming low-income and public HS privileges only, then
  - to any student
- If there are seats available in Q_{MI}, those are made available:
  - to students claiming minority and public HS privileges only, then
  - to students claiming low-income, minority and public HS privileges, then
  - to students claiming low-income and public HS privileges only, then
  - to any student

It is not specified, however, in which order those seats are filled following those priorities[4].

[4] In section 5 we present two actual implementations being used by universities surveyed, clarifying the order in which those seats are filled.

3 The Model

We are dealing with a student-program matching problem where programs have complex privileges structures and students have more than one way to attend a program. Due to those characteristics of the problem we will use the matching with contracts model. There are finite sets $S = \{s_1, \ldots, s_n\}$ and $P = \{p_1, \ldots, p_m\}$ of students and programs. The set $S^P \subseteq S$ contains all students in S from public high-schools, $S^m \subseteq S^P$ contains the racial minority students from public schools and $S^i \subseteq S^P$ contains the low-income students from public schools. Each program p has its own capacity level Q_p and minority reserve ratio r_p^m. Each student s has a vector of exam scores $z(s) = (z_{p_1}(s), \ldots, z_{p_m}(s))$ such that $z_p(s)$ indicates the score of student s for program p. For any two students s and s', $z_p(s)$ and $z_p(s')$ are assumed to be different, that is, $\forall s, s' \in S$ and $p \in P$, $z_p(s) = z_p(s') \iff s = s'$. Each student s has a vector of available privileges she can claim, $t_s = (t_s^P, t_s^m, t_s^i)$ where t_s^P, t_s^m, t_s^i represents public HS, minority and low-income privileges, respectively. Each element of t_s is binary and 1 means student is eligibile for the privilege and 0 means she is not eligible. For example, if a student is a low-income non-minority student from public high school, then $t_s = (1, 0, 1)$. In the Brazilian system, if a student claims public HS, minority or low-income privileges she is required to prove those classifications. Therefore, some students may opt not to claim a privilege associated to a group she belongs to, but students who don't belong to a group (and therefore can't prove belonging to it) are unable to claim that privilege.

Throughout this section we will make use of the *matching with contracts* notation. A contract x, in this context, is a tuple (s, p, t), where $s \in S$, $p \in P$ and $t = (t^P, t^m, t^i) \leq t_s$. Vector t represents the set of privileges student claims and t^P, t^m, t^i are binary and represents public HS, minority and low-income privileges she claims, respectively. For a contract x; x_S, x_P and x_T represent student, program and set of privileges s claims in contract x respectively. Let X be the set of all contracts. For ease of notation, for a set of contracts Y , Y_i is the subset of Y that contains only the contracts that include $i \in S \cup P$. Let $s(Y)$, moreover, be the set of students with contracts in Y, that is, $s(Y) = \{s \in S : \exists (s, p, t) \in Y\}$. An allocation is a set of contracts $X' \subset X$, such that for every $s \in S$ and every $p \in P$, $|X'_s| \leq 1$ and $|X'_p| \leq Q_p$. Let χ be the set of all possible allocations.

The null contract, meaning that the student has no contract, is denoted by $\emptyset$. Students have complete preferences, $\succeq$, over her contracts and the null contract, $X_s \cup \emptyset$. These preferences are derived from students' strict preferences, $\succ^*$, over programs and being unmatched, in addition to the fact that they consider irrelevant how they are accepted to a program:

$$\forall s \in S, \forall p, p' \in P \text{ and } t, t' \leq t_s : (s, p, t) \succ_s (s, p', t') \iff p \succ_s^* p'$$

Next, the choice function of program p, $C_p : 2^X \to 2^X$ is a function that chooses, that is, for $Y \subset X$, $C_p(Y) \subset Y_p$, $C_p(Y)$ has cardinality at most Q_p and has at most one contract for each student. The assumption about student preferences we mentioned above is one of the main differences of our paper with the current matching with contracts literature, since our model allows indifferences among contracts, in contrast with the usual assumption of strict preferences found in the literature so far. Due to indifferences students have between some contracts, we cannot derive choice functions of students as defined in the many to one matching with contracts models. As a result, instead of choice functions for students, we are going to use student preferences. Therefore, primitives of our model are student preferences over contracts and programs' choice functions.

A mechanism is a strategy space Δ_s for each student s along with an outcome function $\psi : \prod_{s \in S} \Delta_s \to \chi$ that selects an allocation for each strategy vector $\prod_{s \in S} \delta_s \in \prod_{s \in S} \Delta_s$. Given a student s and a strategy profile $\delta_s \in \Delta_s$, let δ_{-s} denote the strategy of all students except student s.

4 Student Optimal Stable Mechanism

4.1 The Multidimensional Brazil Privileges Choice Function

One of our objectives is to find a choice function that satisfies the affirmative action objectives for each program, removes incentives for students to strategize over the privileges that they claim and guarantees the existence of a stable allocation. We also aim to design a mechanism that carries out our choice function's properties and finds a stable allocation.

We are proposing a new choice function, Multidimensional Brazil Privileges Choice Function (or MCF), in order to allocate students to seats in programs. Unlike the BRCF, our choice function C^{MCF} obtains the desired incentive characteristics by giving priority in a seat to any student who *can* claim the privileges associated with that seat. Also, by doing this, the choice function satisfies another important criterion: *fairness*.

Let q_p be the number of seats associated with students who claim low-income, minority and public HS in the BRCF, for program p. For any given set of contracts X, the algorithm which implements the choice function C^{MCF} is the following:

Phase 0: Program p rejects each contract that does not include itself ($x_P \neq p \implies x \notin C_p(X)$).

Phase 1: Program p considers only contracts with $x_T = (1, 1, 1)$. Program p accepts contracts including students with the highest scores z_p one at a time and continues until either all contracts are considered or q_p contracts are chosen. In any case, program proceeds with Phase 2. Let θ be $q_p - |\{\text{contracts accepted in Phase1}\}|$.

Phase 2: Program p considers remaining contracts with $x_T > (1, 0, 0)$. Program p accepts contracts including students with highest scores z_p one at a time. During the process, if constraint (1) or (2) below binds, program p tentatively rejects all the remaining contracts with the relevant vector of privileges. Then, the program continues accepting contracts one by one following the order of student scores. Phase 2 ends if all contracts are considered or

$r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} - q_p$ contracts are accepted. Then, the program proceeds with Phase 3.

Possible constraints to bind	Rel. vectors of priv.	
$\lvert\{\text{Contracts accepted with } x_T = (1,0,1)\}\rvert \leq \frac{Q_p}{4} + \theta - q_p$	$t = (1,0,1)$	(1)
$\lvert\{\text{Contracts accepted with } x_T = (1,1,0)\}\rvert \leq r_p^m \frac{Q_p}{2} + \theta - q_p$	$t = (1,1,0)$	(2)

Phase 3: In this phase, the program considers all tentatively rejected contracts and all the remaining contracts with $x_T \geq (1,0,0)$. Program p accepts contracts including students with highest scores z_p, one at a time. The program continues until either all contracts are considered or $\frac{Q_p}{2}$ students are chosen. In any case, it proceeds to Phase 4.

Phase 4: In this phase, the program considers all the remaining contracts. Program p accepts contracts including students with highest scores z_p, one at a time. It continues until either all contracts are considered or Q_P students are chosen. Then program terminates the procedure and rejects all the remaining contracts, if there are any.

4.2 Stability

As in Gale and Shapley (1962) and most of the matching literature, we are interested in stable allocations. Intuitively, an allocation is stable if students or programs cannot improve upon the chosen allocation by either walking away from it or by bilaterally making arrangements outside of the allocation.

Definition 1 *An allocation X' is **stable** if*

i. for all $s \in S$ and for all $p \in P$, $X'_s \succ_s \emptyset$, $C_p(X') = X'_p$; and

ii. $\nexists (p,s) \in P \times S$, and contract $x \in X \setminus X'$, such that
$$x \in C_p((X' \setminus X'_s) \cup \{x\}), x \succ_s X'_s.$$

One can see that if students have strict preferences over contracts then our stability definition and the one used in the current literature would be equivalent. In order to show the existence of a stable allocation, we use the *substitutes* and *law of aggregate demand* properties defined in Hatfield and Milgrom (2005) and *irrelevance of rejected contracts* defined in Aygün and Sönmez (2013).

4.3 Substitutes, IRC, Law of Aggregate Demand and the Student Optimal Stable Mechanism

In this section, we define the properties which are sufficient for existence of a stable allocation in our college admission problem and show that $C^{MCF}()$ has these properties.

Definition 2 *Elements of X are **substitutes** for program p if for all $Y' \subset Y'' \subset X$ we have $x \in Y' \setminus C_p(Y') \implies x \in Y'' \setminus C_p(Y'')$.*

The substitutes condition simply states that if a contract x is rejected, not chosen, in a set of contracts Y' then adding any other contract to Y' cannot make x desirable or x should remain rejected in bigger sets that contain Y'.

Lemma 3 *Elements of X are substitutes for each program p under the choice function C^{MCF}.*

Definition 4 *A choice function C satisfies the **Law of Aggregate Demand** if for all $Y' \subset Y'' \subset X$ we have $|C(Y')| \leq |C(Y'')|$.*

Under the law of aggregate demand, when more contracts are added to a set of contracts, the size of the chosen set never shrinks. Since, in any phase of the choice function unfilled seats are transferred to the next phases, and any student is acceptable to programs, we can state the following lemma.

Lemma 5 *The choice function C^{MCF} satisfies the Law of Aggregate Demand, as defined for each program p.*

For many to one matching problems that use choice functions of programs as a primitive, Aygün and Sönmez (2013) show that the substitutes condition is not sufficient to guarantee existence of stable allocations. Therefore, since our primitive of the model for programs is choice functions rather than preferences, we use the Irrelevance of Rejected Contracts[5] condition defined by Aygün and Sönmez (2013) along with the substitutes condition.

Definition 6 *Given a set of contracts X, a choice function C satisfies the **Irrelevance of Rejected Contracts** (IRC) condition if*

$$\forall Y \subset X, \forall x \in X \setminus Y \quad x \notin C(Y \cup \{x\}) \implies C(Y) = C(Y \cup \{x\}).$$

The IRC condition simply states that an outcome of the choice function should not be affected by the removal of rejected contracts. With the help of this condition, Aygün and Sönmez (2013) show that we can guarantee the existence of stable allocation without the need for strict preferences of programs over sets of contracts.

Lemma 7 *The choice function C^{MCF} satisfies Irrelevance of Rejected Contracts for each program p.*

Finally, with the help of the conditions above, we can guarantee the existence of a stable allocation for our student-program matching problem.

[5] The Irrelevance of Rejected Contracts condition was previously defined as "Consistency" in Alkan and Gale (2001).

Proposition 8 *If all programs use C^{MCF}, the set of stable allocations for student-program matching problem is not empty.*

The choice function defined above defines only how a single school should behave for a given set of students. Now, with the help of that choice function, we are ready to introduce the Student Optimal Stable Mechanism, ψ^{SOSM}. First, students submit a vector of privileges they want to claim and preferences $\succ$. We then use the student proposing cumulative offer algorithm with submitted vector of privileges $(t^s)_{s\in S}$, preferences $\succ$ and C^{MCF} for each program. The cumulative offer algorithm description we use here was previously introduced by Hatfield and Kojima (2010).

Step 1: One randomly selected student s_1 offers her first choice contract x^1 with the vector of privileges (t^{s_1}), according to her preferences $\succ_{s_1}$. The program that receives the offer, $p_1 = x^1_P$, holds the contract. Let $A_{p_1}(1) = x^1$, and $A_p(1) = \emptyset$ for all $p \neq p_1$.

In general,

Step $k \geq 2$: One of the students for whom no contract is currently held by a program, say s_k, offers the most preferred contract with the vector of privileges (t^{s_k}), according to her preferences $\succ_{s_k}$, that has not been rejected in previous steps. Call the new offered contract, x^k. Let $p_k = x^k_P$ hold $C_{p_k}(A_{p_k}(k-1) \cup \{x^k\})$ and reject all other contracts in $A_{p_k}(k-1) \cup \{x^k\}$. Let $A_{p_k}(k) = A_{p_k}(k-1) \cup \{x^k\}$, and $A_p(k) = A_p(k-1)$ for all $p \neq p_k$.

The algorithm terminates when either every student is matched to a program or every unmatched student has no contract left with the vector of privileges they submit to offer. The algorithm terminates in some finite number K of steps due to a finite number of contracts. At that point, the algorithm produces $X' = \bigcup_{p\in P} C_p(A_p(K))$, i.e., the set of contracts that are held by some program at the terminal step K.

We have already shown that the set of stable allocations is not empty if the choice functions satisfy the substitutes condition. Our result below shows that the student optimal stable mechanism gives us a stable allocation which is one of the main desired properties of a mechanism in the matching literature.

Proposition 9 *The Student Optimal Stable Mechanism, ψ^{SOSM}, produces a stable allocation for any given problem.*

4.4 Privilege Monotonicity, Fairness and Affirmative Action Objectives

An ideal choice function should also satisfy Privilege Monotonicity and fairness. Privilege Monotonicity suggests that when a student applies to a program, claiming an additional privilege should not decrease her chance to be chosen. With this property, we can state that for any school, students do not have to gather information and strategize their application processes with respect to those privileges. Hence, we can level the playing field for students.

Definition 10 *Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is* ***Privilege Monotonic*** *if for any given set of contracts $Y \subset X$, and any student s with no contract in Y,*

$$(s, p, t_s) \notin C_p(Y \cup \{(s, p, t_s)\}) \implies (s, p, t') \notin C_p(Y \cup \{(s, p, t')\}), \forall t' \leq t_s.$$

Proposition 11 *The choice function C^{MCF} is Privilege Monotonic.*

Unlike the BRCF, the choice function we design gives students no incentive to leave a privilege, associate to a group she belongs to, unclaimed. This property will have an important role in the strategic properties of the mechanism we suggest.

Definition 12 *Given a set of contracts X, a choice function $C : 2^X \to 2^X$ is* ***fair*** *if for any given subset $Y \subset X$, any program p and $x \in Y_p$,*

$$x \notin C_p(Y) \implies \forall y \in C(Y),\ \text{either } z_p(y_S) > z_p(x_S) \text{ or } x_T \ngeq y_T \geq (1, 0, 0).$$

Fairness of the choice function as we use here indicates that, if a contract is not chosen this means that chosen contracts either include students with higher test scores or they are chosen due to the affirmative action policy.

Proposition 13 *The choice function C^{MCF} is fair.*

The new law issued in Brazil requires some structure on the sets chosen by programs, with respect to the groups to which the students belong to. In other words, the ratios associated with public HS, low-income and minorities should be, when possible, satisfied by the students chosen for each program. We formalize this in the definition below.

Definition 14 *A choice function $C_p : 2^X \to 2^X$* ***satisfies the affirmative action objectives*** *at program p if $\forall Y \subset X$:*

$$\begin{aligned} |\{x &\in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}, \\ |\{x &\in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}, \\ \text{and } |\{x &\in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}. \end{aligned}$$

The definition above states that a choice function must choose a sufficient number of students from all groups of students that are subject to affirmative action, whenever it is possible. One can check that when $q_p = 0$ our choice function satisfies the affirmative action objectives. However, when $q_p = 0$ and $r_p^m = \frac{1}{2}$, all the seats that give priority for those who claim public HS privilege will be reserved only for those who also claim low-income and/or minority privileges. In this case, those who claim only public HS privilege will in practice not have any privilege unless there are not enough applications from those claiming the other combinations of privileges. Also, students claiming all privileges may not enjoy this advantage unless their scores are high enough compared to those claiming only two. The current guidelines set by the Brazilian government give priority to students claiming only public HS privilege for some seats. Due to this fact, one can argue that there is an implicit objective that programs should give priority to all combination of privileges which include public HS for some seats. Since giving priority to each group may cause incentive problems, our choice function, as a second best, prioritizes seats to students who claim each such combination of privileges along with all students who claim some subset of them. For a given program p, let q_p be the number of seats associated with students who claim low-income, minority and public HS in the BRCF. Therefore, if a program p, receives at least q_p contracts with the vector of privileges $(1,1,1)$, the program should accept at least q_p contracts with the vector of privileges $(1,1,1)$. Otherwise, the program should accept all contracts available with the vector of privileges $(1,1,1)$.

Definition 15 *A choice function $C_p : 2^X \to 2^X$ **satisfies the affirmative action objectives conditional on** q_p at program p if $\forall Y \subset X$:*

$$\begin{aligned}
|\{x &\in Y : x_T = (1,1,1)\}| \geq q_p \textit{ implies} \\
|\{x &\in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}, \\
|\{x &\in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}, \\
\textit{and } |\{x &\in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}.
\end{aligned}$$

This second version includes a condition on the number of contracts claiming all privileges. This conditional satisfaction of the affirmative action objectives requires satisfying them only in situations where we have enough applications claiming all three privileges, as well as requiring that the satisfaction of all affirmative action objectives is possible.

Proposition 16 *The choice function C^{MCF} satisfies the affirmative action objectives conditional on q_p at any program p.*

Although depending on the set of contracts available C^{MCF} may not choose a set of contracts that satisfies the affirmative action objectives, q_p can be determined differently for different programs. While programs that set low q_p minimize the number of cases that fail to give enough seats to students claiming certain combinations of privileges, programs that set a higher value for q_p give more opportunity to students who claim only the public HS privilege. One possible way for setting q_p is to construct an expected number of applications claiming all three privileges based on past years' applications.

4.5 Incentives and Fairness of the Student Optimal Stable Mechanism

Although we have shown that the choice function that we proposed satisfies the desired fairness and incentives properties, we are also interested in knowing whether corresponding properties are satisfied by the overall allocation when the SOSM mechanism is used to match students to programs. The first such property that we introduce is that of fairness.

Definition 17 *An allocation X' is **fair** if for any given pair of contracts $x, y \in X'$*

$$y_P \succ^*_{x_S} x_P \implies \text{either } z_{y_P}(y_S) > z_{y_P}(x_S) \text{ or } x_T \not\geq y_T \geq (1,0,0).$$

A mechanism is fair if for any given problem it chooses a fair allocation.

In the previous school choice and student placement literature, like for example in Balinski and Sönmez (1999), it is shown that stability is sufficient for the allocation to satisfy a fairness condition based on the priorities that students have at the schools. This idea comes from the fairness of the responsive preferences of schools. As opposed to the previous school choice and student placement literature, programs in our model do not have responsive preferences. The non existence of responsive preferences may result in allocations that are not fair as in Balinski and Sönmez (1999). Therefore, in our problem, the stability of the mechanism is not sufficient for fairness. That is the reason why the fairness satisfied by our mechanism comes from the fairness of the choice function.

Proposition 18 *The Student Optimal Stable Mechanism, ψ^{SOSM}, is fair.*

The next property that we discuss here is the incentive compatibility of the mechanism, which is a desired characteristic in mechanism design. Incentive compatibility in this context can be described as a property that guarantees that students cannot be better-off by strategizing over manipulations of the preferences being submitted or privileges being claimed. In our problem, students' strategy spaces do not consist only of preferences over schools but also the privileges claimed. Although it is tempting to conclude that the incentive compatibility of the SOSM immediately follows as a corollary of the well-known incentive properties of the SOSM mechanism, due to the wider strategy space for students the result must be obtained explicitly.

Definition 19 *A mechanism is **incentive compatible** if*

$$\nexists s \in S, \delta_{-s} \in \prod_{j \in S \setminus \{s\}} \Delta_j, (t_s, \succ_s), \delta' \in \Delta_s, \text{ such that } \psi(\delta', \delta_{-s}) \succ_s \psi((t_s, \succ_s), \delta_{-s}).$$

In other words, for any student that we consider, no matter what her true preferences are or which groups she belongs to, it will be in her best interest to reveal her true preferences and claim all privileges that she's eligible to. This is valid for any allocation problem and any strategies other students report.

Proposition 20 *The Student Optimal Stable Mechanism, ψ^{SOSM}, is incentive compatible.*

5 Current Mechanism Revisited

So far, we introduced some desired properties that a choice function and a mechanism should satisfy. In this section, first we formally describe two of the choice functions which are implementations of the guidelines published by the Ministry of Education and currently used by two of the largest federal universities in Brazil. Next, we show some deficiencies of those choice functions and any stable mechanism that uses these choice functions.

5.1 Two Examples of the BRCF

Since the specification given by the guideline allows for different choice procedures, we can find variation on the universities' implementation of it. We will describe two instances: the choice function used by the Federal University of Minas Gerais (UFMG) and by the Federal University of Rio Grande do Sul (UFRGS) .

The implementations by UFMG and UFRGS are in the class of choice functions described in Westkamp (2013) and Kominers and Sonmez (2012). This relationship is helpful to analyze our properties.

As we mentioned in section 2, for any program, seats are partitioned into five: Q_{mi}, Q_{Mi}, Q_{mI}, Q_{MI} and Q_{-}. For any given program, numbers of seats and priority structure of Q_{mi}, Q_{Mi}, Q_{mI} and Q_{MI} are determined by the current guideline and are as we discussed in section 2. Since it is not possible to know actual demographic backgrounds of students for the priority structure, both implementations we discussed here takes claims of privileges as demographic backgrounds of students. For any given set of contracts, the choice function used by UFMG, $C^{UFMG}()$, works as the following:

Choice function fills seats in the following order: Q_{mi}, Q_{Mi}, Q_{mI}, Q_{MI} and Q_{-}. For the priorities of the first four group of seats choice function uses priorities described by the current guideline and for the last group, Q_{-}, it gives priority to contracts with privilege vector $(0,0,0)$. If there are seats available in Q_{-} choice function gives priority

- to contracts with privilege vector $(1,1,1)$, then
- to contracts with privilege vector $(1,0,1)$, then
- to contracts with privilege vector $(1,1,0)$, then
- to contracts with privilege vector $(1,0,0)$

During this procedure, choice function either accepts all the contracts or fills all the seats. In any case, choice function stops the procedure and rejects all the remaining contracts, if there is any.

On the other hand, the choice function used by UFRGS, $C^{UFRGS}()$, works as the following:

Choice function fills seats in the following order: Q_{-}, Q_{MI}, Q_{mI}, Q_{Mi} and Q_{mi}. For the priorities of the last four group of seats choice function uses priorities described by the current guideline and for the first group, Q_-, it accepts contracts one at a time based on student scores starting with the contract of student with highest score. During this procedure, choice function either accepts all the contracts or fills all the seats. In any case, choice function stops the procedure and rejects all the remaining contracts, if there is any.

Once we define these two implementations of the BRCF guidelines, the bilateral substitutes property of contracts directly comes from the second proposition of Kominers and Sönmez (2012). Also, since there is only one possible contract for each student to offer to a given program, the choice over contracts satisfies the substitutes condition. Moreover, since each contract is acceptable to all slots, with a bigger contract sets the set of contract chosen never shrinks. Therefore, C^{UFMG} and C^{UFGRS} satisfy the Law of Aggregate Demand. Hence, if all programs use one of the implementations above, the existence of a stable allocation is guaranteed by Proposition 1 of Aygün and Sönmez (2013).

5.2 Two Examples of the BRCF

The two implementations of the guidelines designed by the Brazilian government are instances of choice functions described in Westkamp (2013) and Kominers and Sönmez (2012). Since these choice functions are designed for a single contract for each student, like C^{MCF}, contracts are not only bilateral substitutes, a weak version of substitutes condition, as shown in Kominers and Sönmez (2012) but also substitutes for each program. But these choice functions, unlike C^{MCF}, fail to satisfy the fairness and privilege monotonicity properties. They also don't satisfy the affirmative action objectives conditional on q_p. We show, using examples, how these choice functions violate these three conditions. We start with privilege monotonicity.

Example 21 *[Privilege Monotonicity] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^8\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = (1,0,0)$, $x_T^6 = (1,1,1)$, $x_T^7 = (1,1,0)$ and $x_T^8 = (1,0,1)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. Consider a low-income minority student from public high school $s \notin s(Y)$ with score $z_p(s) > z(x_S^8)$. If she applies with a contract that includes all of her privileges, i.e. $(s,p,(1,1,1))$, no matter which example of the BCRF program p uses, she will be rejected:*

$$(s,p,(1,1,1)) \notin C_p(Y \cup \{(s,p,(1,1,1))\}) = \{x^1,x^2,x^3,x^4,x^5,x^6,x^7,x^8\}$$

However, if she claims only low-income and public HS privileges, i.e. $(s,p,(1,0,1))$, no matter which implementation of BRCF program p uses, her contract will be accepted:

$$(s,p,(1,0,1)) \in C_p(Y \cup \{(s,p,(1,0,1))\}) = \{x^1,x^2,x^3,x^4,x^5,x^6,x^7,(s,p,(1,0,1))\}$$

Therefore, the two examples of the BRCF are not privilege monotonic.

The example above shows that since the choice function gives priority to students who claim low-income and public HS only, the choice function gives student s incentive not to claim her minority privilege. This problem can be solved by using C^{MCF} instead. C^{MCF} gives students equal or higher chances to be chosen when their contracts compete with others that has a subset of the privileges that she claims. Hence students have no incentive not to claim privileges. The second example we give regards the fairness property of choice functions.

Example 22 *[Fairness] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = x_T^6 = (1,1,1)$, $x_T^7 = (1,0,1)$, $x_T^8 = (1,1,0)$ and $x_T^9 = (1,0,0)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. In this case, no matter which example of the BCRF program p uses, the chosen set will be:*

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^7, x^8, x^9\}$$

Let $x_S^6 = j$. Since student j can offer x^6, we can say that $t_j = (1,1,1)$ and $(1,0,0) < t_j$. Also, by assumption, she has higher score than owner of contract x^9. Therefore, rejecting x^6 while accepting x^9, violates fairness of the choice function.

In this second example, the program p chooses x^9, although student j has higher score and claims more privileges than privileges claimed in x^9. This example tells us that the guideline provided by the government implicitly tries to provide diversity in the chosen students even when the law does not require it. On the other hand, C^{MCF} only gives priority to students to which the affirmative action is addressed to. Therefore, C^{MCF} prevents any fairness problems. The next example is about the relationship between choice functions and the affirmative action objectives.

Example 23 *[Affirmative Action conditional on q_p] For a given program p let $Q_p = 8$, $r_p^m = \frac{1}{2}$ and let the set of contracts be $Y = \{x^1, \ldots, x^9\}$ such that $x_T^1 = x_T^2 = x_T^3 = x_T^4 = (0,0,0)$, $x_T^5 = x_T^6 = (1,0,0)$, $x_T^7 = x_T^8 = (1,1,1)$ and $x_T^9 = (1,0,1)$. Also let $z_p(x_S^i) > z_p(x_S^j) \iff i < j$. In both implementations of the BRCF guidelines, the number of seats with priority for students who claim all the 3 privileges is 1 and one seat accepts a contract with privilege vector $(1,0,0)$ since there is no contract claiming minority and public HS privileges only. If the set of contracts is Y, no matter which example of the BRCF program p uses, the chosen set will be:*

$$C_p(Y) = \{x^1, x^2, x^3, x^4, x^5, x^6, x^7, x^9\}$$

Therefore, the choice function chooses only one student claiming minority and public HS privileges, although it is possible to choose two, which is the number of seats with priority for students claiming those privileges.

Another problem with the BRCF is that it considers students claiming public HS privilege only as the first order substitutes for students claiming minority and public HS privileges only. Therefore, when there is an absence of applications from contracts with privilege vector $(1,1,0)$, the choice function turns to contracts with privilege vector $(1,0,0)$ and ignores the priority for minorities. In the example above, one of the students claiming only public HS privilege receives the seat with priority for those claiming minority and public HS privileges. Hence, implementations of the BRCF fail to satisfy the affirmative action objectives conditional on q_p.

Now, we will show that if programs adopt one of the implementations of BRCF above, no matter what algorithm one chooses in order to create a stable mechanism, the mechanism violates the properties we defined above. Previous papers have shown us that some of the deficiencies of choice functions can be corrected by choosing the right algorithm. One example of this is the choice function used by the U.S. Military Academy (USMA). Sönmez and Switzer (2013) have shown us that the USMA priorities may fail to satisfy fairness, but than when they use the cumulative offer algorithm the outcome of the mechanism is always fair. However, the following two examples show that violations of incentive compatibility and fairness are carried by any stable mechanism.

Example 24 *[Incentive Compatibility] There is one program* p *with capacity of eight seats and nine students* $S = \{s_1, \ldots, s_9\}$*. Let* $r_p^m = \frac{1}{2}$ *and* p *be preferred to the null contract by every student. The score order of students is given by* $z_p(s_i) > z_p(s_j) \iff i < j$*. Also, vectors of privileges available to students are given by*

$$t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} = (0,0,0)$$
$$t_{s_5} = t_{s_6} = (1,1,1)$$
$$t_{s_7} = (1,0,0)$$
$$t_{s_8} = (1,1,0)$$
$$t_{s_9} = (1,0,1)$$

For this problem, if every student claims all of the privileges that she is eligible to, there is only one stable allocation, X'*, that we can achieve if program* p *uses one of the implementations of the current BRCF. The set of students assigned is the following:*

$$s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}$$

Now, assuming that the other students use the same strategy as before, if s_6 *claims only public HS privilege and submits* $(s_6, p, (1,0,0))$*, there is again only one stable allocation, say* X''*, that we can achieve if the program* p *uses one of the implementations of the current BRCF and the set of students assigned is the following:*

$$s(X'') = \{s_1, s_2, s_3, s_4, s_5, s_6, s_8, s_9\}$$

Therefore, any stable mechanism with these two examples of the BRCF are not incentive compatible.

The example above shows that since these choice functions give priority to students who claim a subset of the privileges that s_6 is eligible to for some of the seats available, they may give student s_6 an incentive not to claim all of her privileges. This not only puts a burden on students to gather more information about their peers and strategize their behavior in order to get better assignments, but also gives some students an unfair advantage in their college applications. Also, violation of incentive compatibility causes an allocation to be chosen which is actually (with respect to the groups to which the students belong to) unstable. It also makes it harder to observe the effect of this affirmative action policy for future decisions over it. The last example we give relates to the fairness property of mechanisms.

Example 25 *[Fairness] There are one program p with capacity of eight seats and nine students $S = \{s_1, \ldots, s_9\}$. Let $r_p^m = \frac{1}{2}$ and p be preferred to the null contract for each student. The score order of students is given as $z_p(s_i) > z_p(s_j) \iff i < j$. Also, the vectors of privileges available to students are given by*

$$t_{s_1} = t_{s_2} = t_{s_3} = t_{s_4} = (0,0,0)$$
$$t_{s_5} = t_{s_6} = (1,1,1)$$
$$t_{s_7} = (1,0,0)$$
$$t_{s_8} = (1,1,0)$$
$$t_{s_9} = (1,0,1)$$

For this problem, if every student claims all the privileges that they are eligible to, there is only one stable allocation, say X', that we can achieve if the program p uses one of the implementations of the current BRCF and the set of students assigned is the following:

$$s(X') = \{s_1, s_2, s_3, s_4, s_5, s_7, s_8, s_9\}$$

Since student s_6 is eligible to claim all privileges and she has higher score than s_7,s_8 and s_9, rejecting $(s_6, p, (1,1,1))$ while accepting $(s_7, p, (1,0,0))$, violates fairness. This result holds no matter what kind of algorithm we use that gives stable allocation with these two implementations of the BRCF.

6 Concluding Remarks

In this paper, we presented a new market design application of university program-student matching that emerged as result of the affirmative action policy that was designed by the Brazilian government to aid minority and low-income students from public high schools. This problem is particularly interesting in the sense that the freedom of not claiming all of the privileges that a student is eligible to during the application process combines the matching and the adverse selection problems. Due to this fact, we defined the property of privilege monotonicity for choice functions for the first time in this literature.

This paper shows that the current guidelines for designing choice functions for programs have avoidable deficiencies, such as generating unfair allocations and giving sophisticated students an advantage over others by manipulating the system.

We proposed a new choice function, denoted the multidimensional Brazil privileges choice function, that can also be used together with the student optimal stable mechanism to generate student assignments. The choice function is privilege monotonic and fair unlike the current choice functions which are implementations of the guidelines designed by the Brazilian government. Moreover, the mechanism we suggest is incentive compatible, fair and yields a stable allocation for any problem.

With a complex privileges structure like we have in this problem, it is hard to satisfy the affirmative action objectives in all cases. We showed that the current choice functions used by programs in Brazil not only fails to satisfy the affirmative action objectives when they are possible but also fails to satisfy a weaker condition that imposes some restrictions over the population of students applying to a program. On the other hand, the choice function we suggest always satisfies that weaker condition and if the parameters for the choice function is selected correctly, the diversity targets in the programs are reached by our procedure.

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8 Appendix

Proof. [Proof of Lemma 7] For any set of contracts Y and any phase i, let Y_k be set of contracts that is considered in phase k. Think about the procedure:

Phase 1. First observe that $Y_1' \subseteq Y_1''$. If a contract x is not accepted in the first phase then either $x \notin Y_1'$ or we have

$$|\{y \in Y_1' : z_p(y_S) > z_p(x_S)\}| \geq q_p.$$

Therefore, either $x \notin Y_1''$, or $Y' \subseteq Y''$ implies

$$|\{y \in Y_1'' : z_p(y_S) > z_p(x_S)\}| \geq q_p.$$

Hence contract x can not be accepted from Y'' in the first phase as well. So, we have $Y_2' \subseteq Y_2''$.

Phase 2. Let θ' and θ'' be number of unused seats in Phase 1 when we use Y' and Y'', respectively. As $Y_1' \subseteq Y_1''$, we have $\theta' \geq \theta''$. If a contract x is not accepted in the second phase then either $x \notin Y_2'$ which means $x \notin Y_2''$, or we have three cases

Case 1: If $x_T = (1,1,1)$, we have

$$\begin{aligned}
&\min\{|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta' - q_p\} + \\
&\min\{|\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta' - q_p\} + \\
&\quad |\{y \in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
&\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta'' - q_p\} + \\
&\min\{|\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta'' - q_p\} + \\
&\quad |\{y \in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well.

Case 2: If $x_T = (1,1,0)$, we have either

$$\begin{aligned}
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| \geq r_p^m \frac{Q_p}{2} + \theta' - q_p, \text{ or} \\
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| + \\
\min\{|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ s.t. } y_T = (1,0,1)\}|, \frac{q_c}{4} + \theta' - q\} + \\
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p.
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| \geq r_p^m \frac{Q_p}{2} + \theta'' - q_p, \text{ or} \\
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}| + \\
\min\{|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}|, \frac{q_c}{4} + \theta'' - q\} + \\
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well.

Case 3: If $x_T = (1,0,1)$, we have either

$$\begin{aligned}
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| \geq \frac{q_c}{4} + \theta' - q, \text{ or} \\
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| + \\
\min\{|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta' - q_p\} + \\
|\{y &\in Y_2' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta' - 2q_p.
\end{aligned}$$

Therefore, $Y_2' \subseteq Y_2''$ implies

$$\begin{aligned}
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| \geq \frac{q_c}{4} + \theta'' - q, \text{ or} \\
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,0,1)\}| + \\
\min\{|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,0)\}|, r_p^m \frac{Q_p}{2} + \theta'' - q_p\} + \\
|\{y &\in Y_2'' : z_p(y_S) > z_p(x_S) \text{ and } y_T = (1,1,1)\}| \geq r_p^m \frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

as well. Hence, contract x can not be accepted from Y'' in the second phase as well. So any contract x that is not accepted from Y' in Phase 2, is not accepted from Y'' in Phase 2. Moreover, that guarantees $Y_3' \subseteq Y_3''$.

Phase 3. Let θ_1' and θ_1'' be the number of unused seats in Phase 2 when we use Y' and Y'', respectively. As $Y_2' \subseteq Y_2''$, we have $\theta_1' \geq \theta_1''$. If a contract x is not accepted in the third phase then either $x \notin Y_3'$ which means $x \notin Y_3''$, or we have

$$|\{y \in Y_3' : z_p(y_S) > z_p(x_S)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1'.$$

Therefore, $Y_3' \subseteq Y_3''$ implies

$$|\{y \in Y_3'' : z_p(y_S) > z_p(x_S)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta_1''$$

as well. Hence, contract x can not be accepted from Y'' in the third phase as well. So any contract x that is not accepted from Y' in Phase 3, is not accepted from Y'' in Phase 3. Moreover, that guarantees $Y_4' \subseteq Y_4''$.

Phase 4. Let θ_2' and θ_2'' be number of unused seats in Phase 3 when we use Y' and Y'', respectively. As $Y_3' \subseteq Y_3''$, we have $\theta_2' \geq \theta_2''$. If a contract x is not accepted in the fourht phase then we have

$$|\{y \in Y_4' : z_p(y_S) > z_p(x_S)\}| \geq \frac{Q_p}{2} + \theta_2'.$$

Therefore, $Y_4' \subseteq Y_4''$ implies

$$|\{y \in Y_4'' : z_p(y_S) > z_p(x_S)\}| \geq \frac{Q_p}{2} + \theta_2''$$

as well. Hence, contract x can not be accepted from Y'' in the last phase as well. So, any contract x that is not accepted from Y' in Phase 4 is not accepted from Y'' in Phase 4.

A contract x is rejected in set Y' means that x must not be accepted in any phase of the procedure. Above, we showed that for any phase if a contract is not accepted from Y', it can not be accepted from Y''. Therefore, if a contract is rejected from set Y' it must be rejected from set Y''. Hence, contracts are substitutes for any program. ■

Proof. [Proof of Lemma 8] By construction of the choice function $C^{MCF}()$, all contracts of a given student can be rejected from a set only when school reaches full capacity. Hence, the size of the chosen set can never shrink as the set of available contracts grows. ■

Proof. [Proof of Lemma 9] The choice function for any program p satisfies the substitutes condition by Lemma 1 and satisfies the Law of Aggregate Demand by Lemma 2. Hence, Lemma 3 is a corollary of Proposition 1 in Aygun and Sonmez (2013) ■

Proof. [Proof of Proposition 8] To proof this proposition we use a parallel problem where each student s has preference, $\succ_s^{t_s}$, over contracts with t_s and all other contracts are unacceptable for s. The choice function for any program p satisfies the substitutes condition by Lemma 1 and satisfies Irrelevance of Rejected Contracts by Lemma 3. Therefore, as a corollary of Theorem 1 in Aygun and Sonmez (2013), there is a stable allocation for a problem consists of $(\succ_s^{t_s})_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. Let one of possible stable allocations for the parallel problem be X'. We next show that X' is a stable allocation for our original problem consists of $(\succ_s)_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$.

Assume this is not true. Then there exists a student-program pair (s, p) and a contract x such that

$$\begin{aligned} x &\in X \setminus X', x_S = s \text{ and } x_P = p \\ x &\in C_p((X' \setminus X_s') \cup \{x\}) \text{ and } x \succ_s X_s'. \end{aligned}$$

Due to privilege monotonicity property of C_p^{MCF}, we can find a contract y such that

$$\begin{aligned} y &\in X \setminus X', y_S = s, y_P = p \text{ and } y_T = t_s \\ y &\in C_p((X' \setminus X_s') \cup \{y\}) \text{ and } y \succ_s X_s' \end{aligned}$$

which contradicts with the stability of X' for the parallel problem. Hence, X' is a stable allocation for original matching problem consists of $(\succ_s)_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. ■

Proof. [Proof of Proposition 9] Think about five cases:

Case 1: Let $t_s = (1,1,1)$. Assume that her contract, x', such that $x'_T = t_s$, is rejected. Now, we are going to show that another contract of her, x, such that $x_T < t_s$, must be rejected. For a given program p, let $x' = (s,p,t_s)$ and $x = (s,p,t')$ where $t' < t_s$ and let $Y' = Y \cup \{x'\}$ and $Y'' = Y \cup \{x\}$. First, observe that if her contract x' is rejected from set Y', then her contract is not chosen in any phase. Therefore, θ', θ'_1 and θ'_2 are all zero since she is considered in all phases. Assume that she offers contract x instead of x'.

Phase 1: If $t' < (1,1,1)$ then x is not considered in the first phase. Moreover, since her contract x' is rejected from set Y', there are at least q_s contracts in Y with the privilege vector (1,1,1). Therefore, $\theta' = \theta'' = 0$ and $= (Y'_2 \setminus \{x'\}) \subseteq Y''_2$.

Phase 2: Observe that if x is rejected from set Y', then we have

$$\begin{aligned}
\min\{|\{y \;\in\; Y'_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}|, r_p^m\frac{Q_p}{2} + \theta' - q_p\} &+ \\
\min\{|\{y \;\in\; Y'_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta' - q_p\} &+ \\
|\{y \;\in\; Y'_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} - 2q_p &
\end{aligned}$$

If $t' = (1,1,0)$, in the second phase we have either

$$\begin{aligned}
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}| \geq r_p^m\frac{Q_p}{2} + \theta'' - q_p \text{ or} \\
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}| + \\
\min\{|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(j) \text{ and } y_T = (1,0,1)\}|, \frac{Q_p}{4} + \theta'' - q_p\} + \\
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} - 2q_p
\end{aligned}$$

Therefore, x can not be accepted in the second phase. If $t' = (1,0,1)$, in the second phase we have either

$$\begin{aligned}
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}| \geq \frac{Q_p}{4} + \theta'' - q_p \text{ or} \\
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,0,1)\}| + \\
\min\{|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,0)\}|, r_p^m\frac{Q_p}{2} + \theta'' - q_p\} + \\
|\{y \;\in\; Y''_2 : z_p(y_S) > z_p(s) \text{ and } y_T = (1,1,1)\}| \geq r_p^m\frac{Q_p}{2} + \frac{Q_p}{4} + \theta'' - 2q_p
\end{aligned}$$

Therefore, x can not be accepted in the second phase. If $t' \ngeq (1,1,0)$ or $t' \ngeq (1,0,1)$, x will not be considered in the second phase, therefore it cannot be accepted in this phase. Hence, no other available contract of student s can be chosen in this phase. Also, $\theta'_1 = \theta''_1 = 0$ and $(Y'_3 \setminus \{x'\}) \subseteq Y''_3$.

Phase 3: Observe that if x is rejected from set Y', then we have

$$|\{y \in Y'_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r^m_p)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p$$

If $(1,0,0) \leq t' < (1,1,1)$, in the third phase we have

$$|\{y \in Y''_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r^m_p)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p$$

Therefore, x can not be accepted in the third phase. If $t' \ngeq (1,0,0)$, x will not be considered in the third phase, therefore it cannot be accepted in this phase. Hence, no other available contract of student s can be chosen in this phase. Also $\theta'_2 = \theta''_2 = 0$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: First, observe that if x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2}$$

If $t' < (1,1,1)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2}$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s can be chosen.

Case 2: If $t_s = (1,1,0)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1 and 2: If $t' < (1,1,0)$, then x is not considered in the first two phases. So, it can not be accepted in the these phases. Also $\theta'_1 = \theta''_1$ and $(Y'_3 \setminus \{x'\}) \subseteq Y''_3$.

Phase 3: As contract x' is rejected from set Y', we have

$$|\{y \in Y'_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r^m_p)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta'_1$$

If $t' \ngeq (1,0,0)$, then x is not considered in this phase, so it can not be accepted in phase 3. If $t' = (1,0,0)$, then in the third phase we have

$$|\{y \in Y''_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r^m_p)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta''_1$$

Therefore, x can not be accepted in the third phase. Hence, no other available contract of student s is chosen. Also $\theta'_2 = \theta''_2$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $t' < (1,1,0)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 3: If $t_s = (1,0,1)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1 and 2: If $t' < (1,0,1)$, then x is not considered in the first two phases. So, it can not be accepted in the these phases. Also $\theta'_1 = \theta''_1$ and $(Y'_3 \setminus \{x'\}) \subseteq Y''_3$.

Phase 3: As contract x' is rejected from set Y', we have

$$|\{y \in Y'_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta'_1$$

If $t' \ngeq (1,0,0)$, then x is not considered in this phase, so it can not be accepted in phase 3. If $t' = (1,0,0)$, then in the third phase we have

$$|\{y \in Y''_3 : z_p(y_S) > z_p(s)\}| \geq (1 - r_p^m)\frac{Q_p}{2} - \frac{Q_p}{4} + q_p + \theta''_1$$

Therefore, x can not be accepted in the third phase. Hence, no other available contract of student s is chosen. Also $\theta'_2 = \theta''_2$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $t' < (1,0,1)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 4: If $t_s = (1,0,0)$ and her contract x' is rejected we can show that x is not chosen in any phase.

Phase 1,2 and 3: If $t' < (1,0,0)$, then x is not considered in the first three phases. So, it can not be accepted in the these phases. Also $\theta'_2 = \theta''_2$ and $(Y'_4 \setminus \{x'\}) \subseteq Y''_4$.

Phase 4: As contract x is rejected from set Y', then we have

$$|\{y \in Y'_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta'_2$$

If $t' < (1,0,1)$, in the fourth phase we have

$$|\{y \in Y''_4 : z_p(y_S) > z_p(s)\}| \geq \frac{Q_p}{2} + \theta''_2$$

Therefore, x can not be accepted in the fourth phase. Hence, no other available contract of student s is chosen.

Case 5: If $t_s \ngeq (1,0,0)$, then x, like x', is only considered in the last phase and can not be chosen since the set of other contracts considered in this phase are identical for Y' and Y''.

Therefore, $(s,p,t_s) \notin Y'$ guarantees $(s,p,t') \notin Y''$, for any $t' < t_s$. Hence, Choice function is privilege monotonic. ■

Proof. [Proof of Proposition 10] For any arbitrary set of contracts Y, owner of any rejected contract x such that $x_T = (1,1,1)$, has lower score than owners of chosen contracts. So, $x \notin C_p^{MCF}(Y)$ and $x_T = (1,1,1) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$.

For any rejected contract x such that $x_T = (1,0,1)$, the only possible two types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1)$ or $(1,1,0)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,1,0)$ and owners of other chosen contracts have higher scores than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,0,1) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract x such that $x_T = (1,1,0)$, the only possible two types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1)$ or $(1,0,1)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,0,1)$ and owners of other chosen contracts have higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,1,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract x such that $x_T = (1,0,0)$, the only possible types of contracts that is chosen and with lower score than x are contracts with privilege vector $(1,1,1), (1,1,0)$ or $(1,0,1)$. But, since $x_T \ngeq (1,1,1)$, $x_T \ngeq (1,1,0)$, $x_T \ngeq (1,0,1)$ and owners of other chosen contracts have higher score than owner of x, we have $x \notin C_p^{MCF}(Y)$ and $x_T = (1,0,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$.

For any rejected contract such that $x_T \ngeq (1,0,0)$, owners of chosen contracts with privilege vector greater than or equal to $(1,0,0)$ may have lower score than owner of x. Also, owners of other chosen contracts have higher score than owner of x. Therefore, we have $x \notin C_p^{MCF}(Y)$ and $x_T =\ngeq (1,0,0) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$. Hence for any type of contract, $x \notin C_p^{MCF}(Y) \implies \forall y \in C_p^{MCF}(Y), z_p(y_S) > z_p(x_S)$ or $x_T \ngeq y_T \geq (1,0,0)$. ■

Proof. [Proof of Proposition 11] For a given program p and given set of contracts Y, let

$$|\{x \in Y : x_T = (1,1,1)\}| \geq q_p.$$

In the first phase q_p contracts with privilege vector $x_T = (1,1,1)$ will be accepted. In the second phase, a contract will be accepted whenever it is in top $\frac{r_p^m Q_p}{2} - q_p$ among contracts claiming minority and public HS privilege, i.e. $x_T \geq (1,1,0)$, in Y_2. Therefore, in the second phase at least $\frac{r_p^m Q_p}{2} - q_p$ and in total at least $\frac{r_p^m Q_p}{2}$ contracts with $x_T \geq (1,1,0)$ will be accepted, otherwise all contracts with $x_T \geq (1,1,0)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,1,0)\}| \geq \min\{\frac{r_p^m Q_p}{2}, |\{x \in Y : x_T \geq (1,1,0)\}|\}.$$

will be satisfied.

Next, consider contracts with $x_T \geq (1,0,1)$. In the first phase q_p contracts with privilege vector $x_T = (1,1,1)$ will be accepted. In the second phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - q_p$ among contracts claiming low-income and public HS privilege, i.e. $x_T \geq (1,0,1)$, in Y_2. Therefore, in the second phase at least $\frac{Q_p}{4} - q_p$ and in total at least $\frac{Q_p}{4}$ contracts with $x_T \geq (1,0,1)$ will be accepted, otherwise all contracts with $x_T \geq (1,0,1)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,0,1)\}| \geq \min\{\frac{Q_p}{4}, |\{x \in Y : x_T \geq (1,0,1)\}|\}.$$

will be satisfied.

Finally, consider contracts with $x_T \geq (1,0,0)$. In the first two phases $\frac{r_p^m Q_p}{2} + \frac{Q_p}{4} - q_p - \theta_1'$ contracts with with privilege vector $x_T > (1,0,0)$, will be accepted. In the third phase, all the contracts with $x_T = (1,0,0)$ and all the tentatively rejected contracts in phase 2 are considered. In this phase, a contract will be accepted whenever it is in top $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q_p$ among contracts with $x_T \geq (1,0,0)$ in Y_3. Therefore, in the third phase at least $\frac{Q_p}{4} - \frac{r_p^m Q_p}{2} + q_p$ and in total at least $\frac{Q_p}{2}$ contracts with $x_T \geq (1,0,0)$ will be accepted, otherwise all contracts with $x_T \geq (1,0,0)$ will be accepted. Hence,

$$|\{x \in C_p(Y) : x_T \geq (1,0,0)\}| \geq \min\{\frac{Q_p}{2}, |\{x \in Y : x_T \geq (1,0,0)\}|\}.$$

will be satisfied. ■

Proof. [Proof of Proposition 12] The contracts are substitutes for any program p by Lemma 1 and choice functions satisfy IRC condition by Lemma 3. Therefore, as a corollary of Theorem 3 in Hatfield and Milgrom (2005) and Theorem 1 in Aygun and Sonmez (2013), SOSM produces a stable allocation for student preferences for a problem consists of $(\succ_s^{t_s})_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. Moreover, as we showed in the proof of Proposition 1, the stable allocation SOSM produces is also stable for the original problem consists of $(\succ_s^{t_s})_{s\in S}$ and $(C_p^{MCF}())_{p\in P}$. Hence, for any problem, the outcome of SOSM is stable. ■

Proof. [Proof of Proposition 13] Assume that is not true. So, we can find $x, y \in X'$ such that $y_P \succ_{x_S}^* x_P$, $z_{y_P}(y_S) < z_{y_P}(x_S)$ and $x_T > y_T$. Since we have $y_P \succ_{x_S}^* x_P$, there exist a contract x' such that $x' = (x_S, y_P, t_{x_S})$ and $x' \succ_{x_S}^* x$. By the design of cumulative offer algorithm, x' must be offered by x_S and be rejected before the final step K. Therefore, at step K, we have $y, x' \in A_{y_P}(K)$ and $X'_{y_P} = C_{y_P}^{MCF}(A_{y_P}(K))$. Since contracts are substitutes for each program and x' is rejected before the final step K, $x' \notin C_{y_P}^{MCF}(A_{y_P}(K))$ must be true. By fairness condition of choice function

$$x' \notin C_{y_P}^{MCF}(A_{y_P}(K)) \implies z_{y_P}(y_S) > z_{y_P}(x'_S) \text{ or } x_T \nleq y_T$$

a contradiction. Hence ψ^{SOSM}, is fair. ■

Proof. [Proof of Proposition 14] For an arbitrary student s, assume that $\delta' = (t', \succ'_s) \neq (t_s, \succ_s)$. Let her assigned program from $\psi^{SOSM}(\delta', \delta_{-s})$ be p^*. Also, let δ'' be a strategy with privilege vector t' and preference with only contract (s, p^*, t') is acceptable. Since choice functions satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, student s gets same assignment from $\psi^{SOSM}(\delta'', \delta_{-s})$. This part is a corollary of Theorem 10 in Hatfield and Milgrom (2005).

Now, let δ''' be a strategy with privilege vector t_s and preference with only (s, p^*, t_s) is acceptable. Due to privilege monotonicity of choice functions, her assignment from $\psi^{SOSM}(\delta''', \delta_{-s})$ must be (s, p^*, t_s).

Finally, since for any given type profile choice function satisfies substitutes condition by Lemma 1 and Law of Aggregate Demand by Lemma 2, we know that students can not manipulate student optimal stable mechanism by submitting different preferences, i.e. $\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s})$, by Theorem 11 in Hatfield and Milgrom (2005). So we have;

$$\psi^{SOSM}((t_s, \succ_s), \delta_{-s}) \succeq_s \psi^{SOSM}(\delta''', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta'', \delta_{-s}) \succeq_s \psi^{SOSM}(\delta', \delta_{-s})$$

Therefore for any δ',

$$\psi^{SOSM}(\delta', \delta_{-s}) \not\succ_s \psi^{SOSM}((t_s, \succ_s), \delta_{-s})$$

Hence ψ^{SOSM} is incentive compatible. ■